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Real and Complex Asymptotic Symmetries in Quantum Gravity, Irreducible Representations, Polygons, Polyhedra, and the A, D, E Series

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Real and complex asymptotic symmetries in quantum gravity, irreducible representations, polygons, polyhedra, and the A, D, E series

By Patrick J. McCarthy

School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, U.K.

Contents

The Bondi–Metzner–Sachs group B is the common asymptotic group of all asymptotically flat (lorentzian) space-times, and is the best candidate for the universal symmetry group of general relativity. However, in quantum gravity, complexified or euclidean versions of general relativity are frequently considered, and the question arises: Are there similar symmetry groups for these versions of the theory? In this paper it is shown that there are such analogues of B, and a variety of further ones, either real in any signature, or complex. The relationships between these various groups are described. Irreducible unitary representations (IRS) of the complexification $\mathbb{C}B$ of B itself are analysed. It is proved that all induced IRS of $\mathbb{C}B$ arise from IRS of *compact* 'little groups'. It follows that some IRS of $\mathbb{C}B$ are controlled by the IRS of the 'A, D, E' series of finite symmetry groups of regular polygons and polyhedra in ordinary euclidean 3-space. Possible applications to quantum gravity are indicated.

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1. Introduction

In 1939 Wigner published a remarkable paper which laid the foundations of special relativistic quantum mechanics. He chose, as starting points, only the most firmly established experimental facts and theoretical principles, so that his work was entirely free of *ad hoc* assumptions. In particular, no assumption was made that relativistic quantum systems should be described by differential equations. In fact, the set of states of a quantum system was identified with the projective space $P(H)$ of complex straight lines, through the origin, of a complex Hilbert space H. Physical measurements were identified with transition probabilities, given by a 'probability function' $\pi: P(H) \times P(H) \rightarrow [0, 1]$. The relativity principle was expressed not through 'covariance', but as the numerical invariance of all transition probabilities under all Poincaré transformations of affine Minkowski space (translations included).

Among Wigner's results were a complete classification of all relativistic invariant systems in terms of irreducible unitary representations (IRS) of the (covering group of the) Poincaré group in H. These IRS were, in turn, identified with elementary particles, and shown to be parametrized by mass and spin. Especially interesting from the perspective of the present day is that Wigner's work describes, completely explicitly, the set of all possible solutions of all possible (specially) relativistic wave equations (without having to find or solve the equations!) in a form *free* of all constraints. These solution sets are, in fact, precisely the IRS of the (cover of the) Poincaré group.

The Bondi–Metzner–Sachs group (BMS group) B was discovered by Bondi *et al.* (1962) for axisymmetric systems, and by Sachs (1962*a*) for general systems, and is the best candidate for the universal symmetry group of general relativity. As such, it quickly attracted attention as an approach to quantum gravity, or quantum theory with gravity, or the problem of 'internal symmetries' (Sachs 1962*b*; Komar 1965; Newman 1965). With these motivations, a study of IRS of B was started by Sachs (1962*b*), and taken further by Cantoni (1967*a*, *b*).

Wigner's work for special relativity, and the universal property of the BMS group for general relativity, make it reasonable to attempt to lay a similarly firm foundation for quantum gravity, or for quantum theory with gravity, by following through the analogue of Wigner's programme with B replacing the Poincaré group. Some years ago I constructed explicitly the IRS of B for exactly this purpose. This work was based on G. W. Mackey's pioneering work on group representations (see, for example, Mackey 1968, 1978); in particular, an extension to the relevant infinite-dimensional case of his semi-direct product theory. (It is interesting that one source of Mackey's work was his extension of the Stone–von Neumann theorem, which is the cornerstone of quantum mechanics). The role of the IRS of the BMS group B is, however, much less well understood than the role of Wigner's IRS. This paper is the starting point of an attempt to make this role better understood, and to relate the group theoretical approach more closely to other approaches to quantum gravity.

The IRS of B may well already be related to other such approaches. For example, certain of these IRS turned out to be induced (in a sense generalizing Mackey's) from the IRS of the finite symmetry groups of the planar regular polygons or of the platonic solids in ordinary euclidean 3-space. That is, certain IRS of B are induced from the (complex linear) irreducible representations of the cyclic or dihedral groups, or the symmetry groups of the tetrahedron, cube or icosahedron (McCarthy 1973*a*). More precisely, the IRS concerned are actually IRS of the corresponding binary groups. (In

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a finer topology for B, infinite discrete groups and distributions also appear (McCarthy 1975).)

Years later, a completely different approach to quantum gravity by several authors led to a study of gravitational instantons, both for the euclidean and complex versions of general relativity. These instantons are a class of asymptotically flat solutions (or at least, locally so) of the self-dual Einstein equations. In the euclidean case, they play a key role in dominating the path integrals (of a Feynman-like approach to quantum gravity), and are obtained by first isolating the self-dual Einstein equations, and then finding exact solutions to these equations. It turned out that these euclidean instantons were also controlled by the same finite symmetry groups of polygons and platonic solids. In the most recent stages of this work, it has been shown (Kronheimer 1986, 1989*a*, *b*) that the parametrization of the instanton solution spaces (moduli spaces) intimately involves the irreducible representations of these same finite symmetry groups (and not just the groups themselves).

At the outset of this approach to quantum gravity, there was no reason whatever to suspect that these particular finite groups, much less their complex linear irreducible representations, should have anything to do with gravitational instantons (which solve the real nonlinear euclidean self-dual Einstein equations). The instanton solution spaces have been beautifully described (in terms of Brieskorn's semi-universal deformations of singularities arising from the finite groups acting in $\mathbb{C}^2$). However, the description involves constraints; a constraint free description is, as yet, unknown.

The IRs of these finite symmetry groups of polygons and polyhedra are related (McKay 1980) to the series A_n, D_n, and E_n ($=E_6, E_7, E_8$) of simple Lie algebras. This A, D, E series appears unexpectedly in a wide variety of completely different (and, at least apparently, unrelated) areas of mathematics. For example, it appears in singularity theory for functions (Arnol'd 1972, 1981), the theory of quivers in linear algebras (Gabriel 1972), and the theory of caustics (Arnol'd 1984). The two approaches to quantum gravity (via BMS IRs and via euclidean instantons) mentioned above could hardly be more different. The former starts with an infinite-dimensional group (the symmetry group B of the theory) and a Hilbert space only (no equation is postulated or solved), and directly constructs the required data in the form of irreducible BMS representations. The latter starts with the self-dual Einstein equations (no symmetry groups or transformation properties are assumed), and constructs the moduli space of solutions of these equations in a simple, but still constrained, fashion. Nevertheless, each approach turns out to be controlled by complex linear IRs of the same finite symmetry groups of polygons or polyhedra. There are two possible reasons for this. Either the A, D, E series is so all-pervasive that the simultaneous appearance here is merely a coincidence, or the two approaches, in similar contexts (asymptotically flat space-times, quantum gravity) are deeply related.

To try to decide between these reasons, it is evidently essential first to relate the contexts of the two approaches as closely as possible. Gravitational instantons appear in a complexified or euclidean version of general relativity, but BMS IRs have only been investigated in real lorentzian space-times. Are there complex or euclidean analogues of B? If so, how are these new groups related to each other and to B? What are the IRs of these new groups? Quite apart from the possible link with instantons, these questions are of much independent interest. Indeed, their answers are part of a direct and reliable group theoretic approach to quantum gravity.

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It is worth remarking that the finite A, D, E groups ('little groups' of B) only appear in general relativity as a consequence of the infinite dimensionality of the symmetry group B of the theory. The corresponding 'little groups' of the Poincaré group are all of infinite order (they are three-dimensional connected Lie groups); this is a consequence of the fact that, for special relativity, the symmetry group (the Poincaré group) is finite dimensional (it is a 10-dimensional Lie group). Only the additional complexity of general relativity gives rise to the finite groups of polygons and polyhedra.

In this paper, I show that there are complex, euclidean and several other analogues of B; indeed real ones in any signature, and I investigate their IRS, going into particular detail for the complexification $\mathbb{C}B$ of B itself. In §2, earlier variants of B are discussed. In §§3 and 4, a wide variety of analogues of B are constructed and described, and their structures and inter-relationships discussed in detail. In §5, the structure of $\mathbb{C}B$ carefully described, together with the action which specifies it as a semi-direct product. Section 6 treats the theory of IRS of all of these groups in general, and §7 the IRS of $\mathbb{C}B$ in particular. The main reason for the prominence of the A, D, E series for B was that the 'little groups' for the IRS of B are compact (McCarthy 1973a). This is in contrast to the little groups of P, some of which are non-compact (namely, those for zero or negative mass squared); these give rise to the possibility of continuous spins in special relativity (Wigner 1939). The compactness result for B only allows discrete spins (McCarthy 1972a), as observed in nature. It is as though the presence of gravity obstructs the unphysical continuous spins of special relativity. That is, gravity gives a possible explanation for the observed discreteness of elementary particle spins.

In §7 I show that a similar (entirely unexpected) result also holds for $\mathbb{C}B$; all IRS of $\mathbb{C}B$ are induced from compact little groups. This means that the A, D, E series is just as prominent for $\mathbb{C}B$ as it was for B; in fact, the series is, in a sense, more prominent for $\mathbb{C}B$. Section 8 is devoted to some brief remarks about the euclidean analogue EB of B, and §9 exhibits $\mathbb{C}B$ as a 'classical' transformation group of symmetries of the complexified null infinity $\mathbb{C}\mathscr{I}^+$. The paper closes with some brief historical remarks in §10.

The theme of this paper started with an SERC grant application (GR/F23613) by the author in September 1988, followed by a second application in February 1989. This was followed by an earlier (unpublished) version (No. 89A59) of the present paper (McCarthy 1989), the title of which is given in the References below.

2. Previous variants of the BMS group

In the first derivations of the BMS group (Bondi *et al.* 1962; Sachs 1962a), B was found as a transformation pseudo-group (of asymptotic isometries) of the asymptotic region of asymptotically flat lorentzian space-times. However, Penrose (1963, 1974) showed that B could be derived as an actual transformation group of the boundary $\mathscr{I}^+$ (future null infinity) of these space-times. In Penrose's terminology, B is precisely the symmetry group of $\mathscr{I}^+$ in the sense that it is the automorphism group of the 'strong conformal geometry' of $\mathscr{I}^+$. An account of this is given in Penrose & Rindler (1986). The characteristic feature of B is that it contains an infinite-dimensional abelian normal subgroup A of so-called 'supertranslations', defined below.

Now, supertranslations and complex self-dual solutions of Einstein's equations ('complex instantons') are related by the $\mathscr{H}$-space methods of Newman and others

(Newman 1976; Hansen *et al.* 1978; Ko *et al.* 1981). These methods, in effect, perform 'complex supertranslations' on the complexification $\mathbb{C}\mathscr{I}^+$ of $\mathscr{I}^+$ to define 'good cuts' (that is, shearfree cuts of $\mathscr{I}^+$). By restricting attention to complexified supertranslations and shears which are holomorphic in a neighbourhood of $\mathscr{I}^+ \subset \mathbb{C}\mathscr{I}^+$, Newman was able to give a remarkable formula, in the form of a contour integral, for a metric on the resulting space $\mathscr{H}$ of good cuts. Remarkably again, this metric turns out to be a general holomorphic self-dual solution of the complex (vacuum) Einstein equations, or 'complex instanton'. (A brief discussion of this is given by Penrose & Rindler (1986).). It is fascinating to note, in the context of the present paper, that Hitchin (1982) has related Newman's 'good cut' equation (describing a complex instanton) to the Eguchi–Hanson (Eguchi & Hanson 1978) metric, which is the first and simplest euclidean instanton in the class mention above. (See acknowledgements at the end of this paper.) In fact, Hitchin solved the good cut equation for the Eguchi–Hanson metric.

Now, holomorphic complexified shears and supertranslations inevitably develop singularities outside some neighbourhood of $\mathscr{I}^+ \subset \mathbb{C}\mathscr{I}^+$, and this leads to domain problems in defining the complexification $\mathbb{C}B$ of B. For this reason, although a local holomorphic version of $\mathbb{C}B$ is implicit in this $\mathscr{H}$-space work, a global definition of $\mathbb{C}B$ has not, as far as I am aware, been given. Here I propose, rather, to concentrate on B as an abstract group, and to obtain the complexification directly, without requiring holomorphicity. This direct approach leads to a C^∞ rather than a holomorphic complexification, which is globally defined, but no longer closely tied to the contour integral formula mentioned above. However, it enables one to find the relationships between the various real and complex versions of B globally, to study classical global actions, and to investigate Hilbert space IRS for the quantum problem. Thus, in the spirit of Klein's Erlangen programme, the geometries, symmetries and elementary quantum systems can all be studied from a single point of view. It seems likely that these actions and IRS will be closely linked with the other approaches.

Some time ago (McCarthy 1972*b*) I described B in a global form which turns out to be suitable for this global C^∞ complexification. In the same paper, I derived, for pedagogical reasons, a two-dimensional euclidean analogue of the BMS group. A calculation similar to the two-dimensional one shows that there is a four-dimensional euclidean BMS group EB which preserves the asymptotic form of 4-metrics of the form

$$ds^2 = (\delta_{\mu\nu} + h_{\mu\nu})\, dx^\mu\, dx^\nu.$$

Here the x^μ are the usual cartesian coordinates on $\mathbb{R}^4$ ($\mu = 0, 1, 2, 3$ and the summation convention is used), $\delta_{\mu\nu}$ is the Kronecker delta, and

$$h_{\mu\nu} = O(r^{-1}),$$

where the order symbol has its usual meaning, and r is the usual radial coordinate:

$$r^2 = \sum_{\mu=0}^{3} (x^\mu)^2.$$

The details of the derivation of this group EB are given in an earlier (unpublished) version of the present paper (McCarthy 1989). EB turns out to involve 'supertranslations' which are arbitrary C^∞ functions on S^3, in contrast to B, which involves arbitrary C^∞ functions on S^2. Also, Ashtekar & Hansen (1978), in examining both spacelike and null infinity for real lorentzian space-times, derived a BMS-like group based on the unit spacelike hyperboloid (see also acknowledgements below). Like EB, this group also involves arbitrary C^∞ functions of three variables; indeed, on the hyperboloid. The next section is devoted to a unified description of these and other groups of similar type, together with their complexifications, by means of a general construction extending the construction of B given in McCarthy (1972*b*).

3. Real and complex BMS groups

(*a*) *Real groups*

Let $\mathbb{R}^{p,q}$ denote the real vector space with scalar product of signature (p,q) for $p+q=4$ (with p pluses and q minuses), and $L(p,q)$ the corresponding 'Lorentz group' $SO_0(p,q)$ of linear transformations preserving the scalar product. (Throughout, I consider only identity components of continuous groups; space or time reflection are excluded.) Thus $\mathbb{R}^{4,0}$ is euclidean space, $\mathbb{R}^{3,1}$ is Minkowski space, and $\mathbb{R}^{2,2}$ is the so-called ultrahyperbolic space, with scalar products

$$\mathbb{R}^{4,0}: \quad x\cdot y = x^0y^0+x^1y^1+x^2y^2+x^3y^3,$$
$$\mathbb{R}^{3,1}: \quad x\cdot y = x^0y^0-x^1y^1-x^2y^2-x^3y^3,$$
$$\mathbb{R}^{2,2}: \quad x\cdot y = x^0y^0+x^2y^2-x^1y^1-x^3y^3,$$

these particular expressions being convenient here. Matrices $\Lambda\in L(p,q)$ are taken as acting from the right on row vectors $x\in\mathbb{R}^{p,q}$. Let K denote any subset of $\mathbb{R}^{p,q}$ invariant under $L(p,q)$ and also under the dilatation action

$$\mathbb{R}^*\times\mathbb{R}^{p,q}\to\mathbb{R}^{p,q}; \quad (t,x)\mapsto tx$$

of the multiplicative group $\mathbb{R}^*$ of positive real numbers. Let $A(K)$ be the vector space (under pointwise addition) of all C^∞ functions $f:K\to\mathbb{R}$ satisfying the homogeneity condition

$$f(tx) = tf(x), \quad \text{all} \quad t\in\mathbb{R}^*.$$

Define a representation T of $L(p,q)$ on $A(K)$ by setting, for each $\Lambda\in L(p,q)$,

$$(T(\Lambda)f)(x) = f(x\Lambda).$$

Now let $B^{p,q}(K)$ be the semi-direct product

$$B^{p,q}(K) = A(K)\circledS_T L(p,q).$$

Thus the group law for pairs in $A(K)\times L(p,q)$ is

$$(f_1,\Lambda_1)(f_2,\Lambda_2) = (f_1+T(\Lambda_1)f_2, \Lambda_1\Lambda_2).$$

The normal subgroup $A(K)$ is, by analogy, called the supertranslation subgroup.

If K is the future pointing null cone in Minkowski space, defined by

$$N^+ = \{x\in\mathbb{R}^{3,1}\,|\,x\cdot x=0, x^0>0\},$$

then $B^{3,1}(N^+)$ is the original BMS group B, since the above construction then specializes to the one given in McCarthy (1972*b*). If K is the set S of all spacelike vectors, defined by

$$S = \{x\in\mathbb{R}^{3,1}\,|\,x\cdot x<0\},$$

then $B^{3,1}(S)$ is easily shown to be isomorphic to the group defined by Ashtekar & Hansen (1978). If, in euclidean space, K is the set

$$\mathbb{R}^4 - 0 = \{x \in \mathbb{R}^{4,0} \mid x \neq 0\},$$

then $B^{4,0}(\mathbb{R}^4 - 0)$ is isomorphic to the four-dimensional euclidean BMS group EB mentioned above, and derived in McCarthy (1989). Thus the construction of the previous paragraph subsumes all real groups of BMS type which have so far been derived.

However, it is considerably more general. Indeed, any K invariant under $L(p,q)$ and $\mathbb{R}^*$ is a disjoint union of orbits of $\mathbb{R}^* \times L(p,q)$ in $\mathbb{R}^{p,q}$. These orbits are easily classified in the various cases. In all cases, the origin $\{0\}$ is an orbit, but since $A(\{0\})$ is just the zero function,

$$B^{p,q}(\{0\}) \simeq L(p,q).$$

That is, there are no supertranslations in this case; it is henceforth discarded as trivial. For $\mathbb{R}^{4,0}$, then, there is only one orbit, $\mathbb{R}^4 - 0$, giving

$$EB \simeq B^{4,0}(\mathbb{R}^4 - 0) = A(\mathbb{R}^4 - 0) \circledS_T L(4,0).$$

For $\mathbb{R}^{3,1}$ there are five orbits, namely

$$T^+ = \{x \in \mathbb{R}^{3,1} \mid x \cdot x > 0, x^0 > 0\},$$
$$T^- = \{x \in \mathbb{R}^{3,1} \mid x \cdot x > 0, x^0 < 0\},$$
$$N^+ = \{x \in \mathbb{R}^{3,1} \mid x \cdot x = 0, x^0 > 0\},$$
$$N^- = \{x \in \mathbb{R}^{3,1} \mid x \cdot x = 0, x^0 < 0\},$$
$$S = \{x \in \mathbb{R}^{3,1} \mid x \cdot x < 0\}.$$

For $\mathbb{R}^{2,2}$ there are three orbits, namely

$$T = \{x \in \mathbb{R}^{2,2} \mid x \cdot x > 0\},$$
$$N = \{x \in \mathbb{R}^{2,2} \mid x \cdot x = 0\},$$
$$\Sigma = \{x \in \mathbb{R}^{2,2} \mid x \cdot x < 0\}.$$

In each case, then, K can be any disjoint union of these orbits.

Evidently, for each signature, the collection of possible Ks is partly ordered by inclusion maps, and there are corresponding restriction maps for the $A(K)$. So, if K is a disjoint union $K = L \cup M$, the inclusion

$$L \hookrightarrow L \cup M$$

corresponds to restricting functions $f \in A(L \cup M)$ to the subset L, giving a map

$$A(L \cup M) \rightarrow A(L)$$

which, in general, is not onto. Correspondingly we have a map

$$B^{p,q}(L \cup M) \rightarrow B^{p,q}(L).$$

In particular, in any signature, we may take for K the union $\mathbb{R}^4 - 0$ of all orbits. It is appropriate to call the resulting groups *universal*, since each of them sits at the top of a lattice like structure defined by the restriction maps. I write, for these universal groups,

$$B^{p,q}(\mathbb{R}^4 - 0) = \mathcal{U}B^{p,q}.$$

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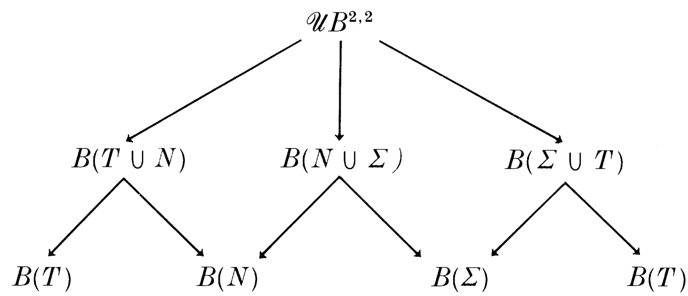
Since, for $\mathbb{R}^{4,0}$, this is the only group, it is already universal;

$$EB = B^{4,0}(\mathbb{R}^4 - 0) = \mathscr{U}B^{4,0}.$$

For the ultrahyperbolic space $\mathbb{R}^{2,2}$, we have

$$\mathbb{R}^4 - 0 = T \cup N \cup \Sigma,$$

and hence the diagram (dropping the superscripts from the lower groups)



which involves seven groups in all (one appears twice in the bottom row). The diagram for $\mathscr{U}B^{3,1}$ may similarly be obtained from the disjoint union

$$\mathbb{R}^4 - 0 = T^+ \cup N^+ \cup S \cup N^- \cup T^-.$$

There are, in all, 31 groups of type $B^{3,1}(K)$, with a corresponding diagram containing $\mathscr{U}B^{3,1}$ at the top.

If K is a single orbit, it is appropriate to call $B^{p,q}(K)$ *atomic*, otherwise *compound*. For all three signatures, then, there is a total of 39 groups $B^{p,q}(K)$. Nine of these groups (including the original B) are atomic, and 30 are compound. Three of them are universal (one for each signature). Only one, EB, is both universal and atomic.

Each of the 39 groups has an infinite-dimensional $A(K)$, and each contains the 'Poincaré group' of appropriate signature. Indeed, restricting $A(K)$ to the four-parameter family of linear functions of the form $f_v(x) = x \cdot v$ where $v \in \mathbb{R}^4$ is a constant vector and dot the appropriate scalar product, the $L(p,q)$ action on $A(K)$ specializes to an action corresponding to $(\Lambda, v) \mapsto \Lambda v$ on $\mathbb{R}^4$. This gives the Poincaré group

$$P^{p,q} = \mathbb{R}^4 \circledS_T L(p,q)$$

in the signature concerned. That is, specializing the nonlinear functions $f \in A(K)$ to linear ones gives the group of special relativity. Thus we may say that 'linearizing the (general relativity) BMS groups gives the (special relativity) Poincaré groups'.

It should be noted that, in the original physical context of asymptotic symmetry groups (gravitational radiation), those new groups above which involve timelike orbits are unlikely to be of direct physical significance (they would presumably refer to tachyonic radiating matter). However, they are likely to be of indirect significance, probably via their complexifications (see next subsection).

(b) *Complex groups*

The above construction can now be complexified as follows. First note that, if the coordinates x^μ are now all taken complex, the various scalar products mentioned above all become equivalent (under the maps obtained by multiplying coordinates by i). The complex 'Lorentz groups' preserving any one of these scalar products are all isomorphic, for the same reason. So we may replace $\mathbb{R}^{p,q}$ by $\mathbb{C}^4$ and $L(p,q)$ by $\mathbb{C}L$ this being the complex group preserving (for example)

$$x \cdot y = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$$

with the x^μ all complex.

Now let $\mathscr{K}$ be any subset of $\mathbb{C}^4$, apart from the origin, invariant under both $\mathbb{C}L$ and also the action

$$\mathbb{C}^* \times \mathbb{C}^4 \to \mathbb{C}^4; \quad (t,x) \mapsto tx$$

of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. Let $\mathbb{C}A(\mathscr{K})$ be the vector space, under pointwise addition, of all C^∞ functions $f:\mathscr{K} \to \mathbb{C}$ satisfying the homogeneity condition

$$f(tx) = tf(x), \quad \text{all} \quad t \in \mathbb{C}^*.$$

Here C^∞ means that, considered as a map on the real spaces (of twice the dimension) underlying $\mathscr{K}$ and $\mathbb{C}$, f is C^∞ in the usual real sense. Define a representation T of $\mathbb{C}L$ on $\mathbb{C}A$ by setting, for each $\Lambda \in \mathbb{C}L$,

$$(T(\Lambda)f)(x) = f(x\Lambda).$$

Now let $\mathbb{C}B(\mathscr{K})$ be the semi-direct product

$$\mathbb{C}B(\mathscr{K}) = \mathbb{C}A(\mathscr{K}) \circledS_T \mathbb{C}\mathscr{L}.$$

Groups of this form are the required complex BMS groups.

In this context, there are only two orbits of $\mathbb{C}^* \times \mathbb{C}L$, namely

$$\mathscr{N} = \{x \in \mathbb{C}^4 \mid x \cdot x = 0, x \neq 0\},$$
$$\mathscr{M} = \{x \in \mathbb{C}^4 \mid x \cdot x \neq 0\}.$$

Hence there are exactly three complex groups of BMS type, namely

$$\mathbb{C}B(\mathscr{N}), \quad \mathbb{C}B(\mathscr{M}), \quad \mathbb{C}B(\mathbb{C}^4 - 0).$$

Since $\mathbb{C}^4 - 0 = \mathscr{N} \cup \mathscr{M}$, it is appropriate to call the third group universal; I write

$$\mathscr{U}\mathbb{C}B = \mathbb{C}B(\mathbb{C}^4 - 0).$$

The first two groups are atomic, and the restriction diagram is very simple:

$$\begin{array}{ccc} & \mathscr{U}\mathbb{C}B & \\ \swarrow & & \searrow \\ \mathbb{C}B(\mathscr{N}) & & \mathbb{C}B(\mathscr{M}). \end{array}$$

Evidently $\mathscr{U}\mathbb{C}B$, which is compound, is the complexification of each real universal group. The atomic group $\mathbb{C}B(\mathscr{N})$ is the complexification of each of the groups $B^{3,1}(N^+)$, $B^{3,1}(N^-)$, and $B^{2,2}(N)$. Finally $\mathbb{C}B(\mathscr{M})$ is the complexification of each of the groups $B^{3,1}(T^+)$, $B^{3,1}(T^-)$, $B^{3,1}(S)$, $B^{2,2}(T)$ and $B^{2,2}(\Sigma)$.

The obvious complex version of the linearization restriction of $B^{p,q}(K)$ to $P^{p,q}$ gives, for any $\mathbb{C}B(\mathscr{K})$, the complex Poincaré group $\mathbb{C}P$ defined by

$$\mathbb{C}P = \mathbb{C}^4 \circledS_T \mathbb{C}L.$$

It is interesting to note that, for special relativity, all Poincaré groups are universal in the following sense. There is exactly one complex Poincaré group, and, in each real signature, exactly one real form for this group. This section shows that the position is much more intricate for the asymptotic symmetry groups of general relativity.

4. Other models for the groups

(a) *Projective spaces*

The orbits of the dilatation action of $\mathbb{R}^*$ on $\mathbb{R}^{p,q}$ are open half-lines from the origin in $\mathbb{R}^{p,q} = \mathbb{R}^4$. For any K, let $P_+(K)$ be the projective-like space of all such half-lines lying in K. Now $A(K)$ is the set of all C^∞ functions $f: K \to \mathbb{R}$ which are homogeneous of degree 1; we may write

$$A(K) = C_1^\infty(K, \mathbb{R}).$$

Since the homogeneity constraint fixes the behaviour of these functions along half-lines, $A(K)$ may also be realized as the set of all arbitrary (unconstrained or 'free') C^∞ functions on $P_+(K)$:

$$A(K) = C_1^\infty(K, \mathbb{R}) \simeq C^\infty(P_+(K), \mathbb{R}).$$

The homeomorphism types of the various $P_+(K)$ are easily found as follows. Every half-line cuts the 3-sphere S^3 given by

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 2$$

exactly once, so we have

$$P_+(K) \simeq S^3 \cap K.$$

In particular, $P_+(\mathbb{R}^4 - 0) \simeq S^3$, so in each signature the universal groups can be realized as

$$\mathscr{U}B^{p,q} \simeq C^\infty(S^3, \mathbb{R}) ⓢ_T L(p,q).$$

Here and below, the actions (also called T) of $L(p,q)$ on the spaces of free functions are easily found, and will be given in detail for $\mathbb{C}B$. Using the definitions of the various orbits K, one easily finds the following table of homeomorphism types of $P_+(K) \simeq S^3 \cap K$:

K	T^+ or T^-	S	N^+ or N^-	T or Σ	N
$P_+(K)$	D^3	$S^2 \times I$	S^2	$S^1 \times D^2$	$S^1 \times S^1$.

Here D^{n+1} is the interior of a solid n-sphere S^n (so $D^{n+1} \simeq \mathbb{R}^{n+1}$) and I is an open interval of $\mathbb{R}$. If K is a disjoint union of orbits, $K = L \cup M$, then we have a corresponding disjoint union for the half-line spaces; $P_+(K) = P_+(L) \cup P_+(M)$. This, then, realizes all of the real groups with the supertranslations now unconstrained;

$$B^{p,q}(K) = C^\infty(P_+(K), \mathbb{R}) ⓢ_T L(p,q).$$

The following cases of this realization are of special interest;

$$B = B^{3,1}(N^+) \simeq C^\infty(S^2, \mathbb{R}) ⓢ_T L(3,1),$$
$$EB = B^{4,0}(\mathbb{R}^4 - 0) \simeq C^\infty(S^3, \mathbb{R}) ⓢ_T L(4,0),$$
$$B^{3,1}(S) \simeq C^\infty(S^2 \times I, \mathbb{R}) ⓢ_T L(3,1),$$
$$HB = B^{2,2}(N) \simeq C^\infty(S^1 \times S^1, \mathbb{R}) ⓢ_T L(2,2).$$

Indeed, the first three are the groups mentioned in §2. The last one, here called HB (for 'hyperbolic BMS group') is, like B itself, based on a null cone, and it has the same complexification $\mathbb{C}B(\mathscr{N})$ as B itself. It is also interesting to note that the complexifications of the first three groups give all of the complex BMS groups, namely $\mathbb{C}B(\mathscr{N})$, $\mathscr{U}\mathbb{C}B = \mathbb{C}B(\mathscr{M} \cup \mathscr{N})$ and $\mathbb{C}B(\mathscr{M})$.

In a similar way, the complex supertranslation groups $\mathbb{C}A(\mathscr{K})$ of C^∞ homogeneous functions $f:\mathscr{K}\rightarrow\mathbb{C}$ may be realized as the set of all (free) C^∞ functions on the corresponding complex projective space $P(\mathscr{K})$ of $\mathbb{C}^*$ orbits in $\mathscr{K}$. That is, $P(\mathscr{K})$ is the space of all complex straight lines through the origin which lie in $\mathscr{K}$. And we have

$$\mathbb{C}A(\mathscr{K}) = C_1^\infty(\mathscr{K},\mathbb{C}) \simeq C^\infty(P(\mathscr{K}),\mathbb{C}).$$

This realizes the three complex groups of §3 as

$$\mathscr{U}\mathbb{C}B \simeq C^\infty(P_3(\mathbb{C}),\mathbb{C})\, \circledS_T\, \mathbb{C}L,$$
$$\mathbb{C}B(\mathscr{M}) \simeq C^\infty(P(\mathscr{M}),\mathbb{C})\, \circledS_T\, \mathbb{C}L,$$
$$\mathbb{C}B(\mathscr{N}) \simeq C^\infty(P(\mathscr{N}),\mathbb{C})\, \circledS_T\, \mathbb{C}L.$$

(*b*) *Double covers*

In quantum mechanics, groups initially act in the projective Hilbert space $P(H)$ of quantum states, so their representations are projective rather than linear (Wigner 1939). In lifting to H itself, one is forced to pass to the double cover P_{c} of the Poincaré group P. In fact, this is the real reason that spinors appear in relativistic quantum mechanics. Correspondingly, for B, it is really unitary representations of the double cover B_{c} (McCarthy 1973*a*, 1975) which are relevant in quantum mechanics. Also, the structure of the groups themselves is most simply described via these double covers.

First I recall the definition of $\mathbb{C}L_{\mathrm{c}}$. Let $M(2,\mathbb{C})$ be the set of all 2×2 complex matrices, and let

$$SL(2,\mathbb{C}) = \{g\in M(2,\mathbb{C}) \mid \det g = 1\}.$$

$SL(2,\mathbb{C})$ is a matrix group, sometimes denoted G below. Define a right action of $G^2 = G\times G$ on $M(2,\mathbb{C})$ by $M(2,\mathbb{C})\times G^2 \rightarrow M(2,\mathbb{C})$ with

$$(m,(g,h)) \mapsto g^{\mathrm{T}} m h,$$

where the superscript T means transpose. Consider the map $s:\mathbb{C}^4\rightarrow M(2,\mathbb{C})$ defined by

$$s(x) = \begin{bmatrix} x^0 - x^3 & x^1 + \mathrm{i}x^2 \\ x^1 - \mathrm{i}x^2 & x^0 + x^3 \end{bmatrix},$$

where the x^μ are the components of $x\in\mathbb{C}^4$. This map is a linear bijection, so the right action of G^2 on $M(2,\mathbb{C})$ induces a linear right action of G^2 on $\mathbb{C}^4$. Since

$$\det(s(x)) = x\cdot x$$

and the G^2 action preserves determinants (indeed $\det g = \det h = 1$) in $M(2,\mathbb{C})$, G^2 acts as linear isometries on $\mathbb{C}^4$, that is, as transformations from $\mathbb{C}L$. In fact, this construction gives a homomorphism

$$\gamma: G\times G \rightarrow \mathbb{C}L,$$

which in onto, and has kernel $\{(Id, Id), (-Id, -Id)\}$ in $G\times G$. Thus γ identifies $G\times G$ as the double cover of $\mathbb{C}L$

$$G\times G = \mathbb{C}L_{\mathrm{c}}$$

and in fact $G\times G$ is the universal cover.

The covering groups of the various real forms of $\mathbb{C}L$ are easily described by making appropriate restrictions. For $L(4,0)$, take x^0 real and x^1, x^2, x^3 pure imaginary, and restrict both g and h to be arbitrary unitary matrices. For $L(3,1)$, take all the x^μ to be real, and restrict (g,h) to pairs of the form $(\bar{h},h)$ for arbitrary $h\in G$. For $L(2,2)$, take x^0, x^1, x^3 real and x^2 pure imaginary, and restrict g and h to be arbitrary real matrices. This gives the double covers of the $L(p,q)$;

$$SU(2)\times SU(2) = SU(2)^2 = L(4,0)_c,$$
$$SL(2,\mathbb{C}) = G = L(3,1)_c,$$
$$SL(2,\mathbb{R})\times SL(2,\mathbb{R}) = SL(2,\mathbb{R})^2 = L(2,2)_c,$$

where $SU(2)$ ($SL(2,\mathbb{R})$) is the subgroup of G of unitary (respectively, real) matrices.

Letting the appropriate groups act via the covering map γ composed with T, we get the corresponding covering groups for the BMS-type groups. The results may be summarized in

Theorem 4.1. *The covering groups of the groups given in* §3 *have the form*

$$\begin{aligned} B^{p,q}(K)_c &= C_1^\infty(K,\mathbb{R})\,Ⓢ_T\,L(p,q)_c \\ &= C^\infty(P_+(K),\mathbb{R})\,Ⓢ_T\,L(p,q)_c, \\ \mathbb{C}B(\mathscr{K})_c &= C_1^\infty(\mathscr{K},\mathbb{C})\,Ⓢ_T\,\mathbb{C}L_c \\ &= C^\infty(P(\mathscr{K}),\mathbb{C})\,Ⓢ_T\,\mathbb{C}L_c, \end{aligned}$$

where $L(p,q)_c$ and $\mathbb{C}L$ are as above.

Strictly speaking, in the theorem, 'T' should read '$T\gamma$', but the notation is simpler as above.

In particular, the theorem gives the following special cases:

$$\begin{aligned} \mathscr{U}\mathbb{C}B &= C^\infty(P_3(\mathbb{C}),\mathbb{C})\,Ⓢ_T\,(G\times G), \\ \mathscr{U}B^{p,q} &= C^\infty(S^3,\mathbb{R})\,Ⓢ_T\,L(p,q)_c, \\ B_c = B^{3,1}(N^+)_c &= C^\infty(S^2,\mathbb{R})\,Ⓢ_T\,G, \\ EB_c = B^{4,0}(\mathbb{R}^4-0)_c &= C^\infty(S^3,\mathbb{R})\,Ⓢ_T\,(SU(2)\times SU(2)), \\ B^{3,1}(S)_c &= C^\infty(S^2\times I,\mathbb{R})\,Ⓢ_T\,G, \\ HB_c = B^{2,2}(N)_c &= C^\infty(S^1\times S^1,\mathbb{R})\,Ⓢ_T\,(SL(2,\mathbb{R})\times SL(2,\mathbb{R})). \end{aligned}$$

The third group in this list is now in the form in which representations were investigated in McCarthy (1973*a*, 1975). Also, we have

$$\mathbb{C}B(\mathscr{N})_c = C^\infty(P(\mathscr{N}),\mathbb{C})\,Ⓢ_T\,(G\times G)$$

for the complexification of B_c (or of HB_c); this complex group is described in detail in the next section.

5. A close-up for $\mathbb{C}B(\mathscr{N})_c$

Since this group is especially important for quantum gravity, and for the possible links to other approaches, I here go into considerable detail in describing the structure. The first part of this section is essentially identical to §3 of McCarthy (1989).

(a) *The complex projective null cone*

The complex null cone $\mathscr{N}$ may be identified with its image in under the linear bijection s of §4;

$$\mathscr{N} = \{m\in M(2,\mathbb{C})\,|\,m\neq 0, \det m = 0\}.$$

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Hence $m \in \mathcal{N}$ if and only if m has rank exactly 1. Set $\mathcal{S}$ be the set of all non-zero complex two-component row vectors; $\mathcal{S} = \mathbb{C}^2 - 0$. In Penrose's terminology, $\mathcal{S}$ is 'spin space'. From the rank condition, it follows that $m \in \mathcal{N}$ if and only if

$$m = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}$$

for some pair $(\boldsymbol{z}, \boldsymbol{w}) \in \mathcal{S} \times \mathcal{S} = \mathcal{S}^2$. Let π be the projection $\pi : \mathcal{S}^2 \rightarrow \mathcal{N}$ given by

$$\pi(\boldsymbol{z}, \boldsymbol{w}) = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}.$$

Then the following result gives $\mathcal{N}$ in terms of $\mathcal{S}^2$.

Proposition 5.1. *Define a left action* $\mathbb{C}^* \times \mathcal{S}^2 \rightarrow \mathcal{S}^2$ *by*

$$(\lambda, (\boldsymbol{z}, \boldsymbol{w})) \mapsto (\lambda \boldsymbol{z}, \lambda^{-1} \boldsymbol{w}).$$

Then this $\mathbb{C}^*$ *action is fixed point free, and the* $\mathbb{C}^*$ *orbits are precisely the fibres of the projection*

$$\pi : \mathcal{S}^2 \rightarrow \mathcal{N}.$$

Thus we have the identification

$$\mathcal{N} \simeq \mathbb{C}^* \backslash (\mathcal{S} \times \mathcal{S}).$$

In other words, the map π *defined by* $\pi(\boldsymbol{z}, \boldsymbol{w}) = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}$ *concretely realizes the principal bundle* $\pi : \mathcal{S}^2 \rightarrow \mathcal{N}$ *with structure group* $\mathbb{C}^*$.

Proof. If $(\lambda \boldsymbol{z}, \lambda^{-1} \boldsymbol{w}) = (\boldsymbol{z}, \boldsymbol{w})$, then in particular, $z_1 \lambda = z_1$ and $z_2 \lambda = z_2$, where z_1 and z_2 are the components of $\boldsymbol{z}$. But z_1 and z_2 do not both vanish, so $\lambda = 1$ and the action is fixed point free. If $(\boldsymbol{z}', \boldsymbol{w}')$ and $(\boldsymbol{z}, \boldsymbol{w})$ belong to the same orbit, $\boldsymbol{z}' = \lambda \boldsymbol{z}$ and $\boldsymbol{w}' = \lambda^{-1} \boldsymbol{w}$ for some $\lambda \in \mathbb{C}^*$, so $\pi(\boldsymbol{z}', \boldsymbol{w}') = \boldsymbol{z}'^{\mathrm{T}} \boldsymbol{w}' = \boldsymbol{z}^{\mathrm{T}} \lambda \lambda^{-1} \boldsymbol{w} = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w} = \pi(\boldsymbol{z}, \boldsymbol{w})$, so the pairs belong to the same fibre. If the pairs belong to the same fibre, $\pi(\boldsymbol{z}', \boldsymbol{w}') = \pi(\boldsymbol{z}, \boldsymbol{w})$ gives

$$\boldsymbol{z}'^{\mathrm{T}} \boldsymbol{w}' = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}.$$

Post-multiplying by the hermitian conjugate $\boldsymbol{w}^*$, of $\boldsymbol{w}$ gives $\boldsymbol{z}'^{\mathrm{T}}(\boldsymbol{w}' \boldsymbol{w}^*) = \boldsymbol{z}^{\mathrm{T}}(\boldsymbol{w} \boldsymbol{w}^*)$. But $\boldsymbol{w} \neq \boldsymbol{0}$, so the real norm $(\boldsymbol{w} \boldsymbol{w}^*) \neq 0$, and since $\boldsymbol{z}^{\mathrm{T}} \neq \boldsymbol{0}$, the number $(\boldsymbol{w}' \boldsymbol{w}^*)$ cannot be zero. Hence $\boldsymbol{z}'$ is a non-zero complex multiple of $\boldsymbol{z}$, $\boldsymbol{z}' = \lambda \boldsymbol{z}$. Taking the transpose of the equation gives, similarly, $\boldsymbol{w}' = \mu \boldsymbol{w}$, $\mu \neq 0$. Putting these into the equation then gives $\lambda \mu = 1$, so $(\boldsymbol{z}', \boldsymbol{w}') = (\lambda \boldsymbol{z}, \lambda^{-1} \boldsymbol{w})$ and the pairs belong to the same orbit. This completes the proof.

On the other hand, the complex projective null cone $P(\mathcal{N})$ is the space of orbits of the action

$$\mathbb{C}^* \times \mathcal{N} \rightarrow \mathcal{N}; \quad (t, n) \mapsto tn$$

and so we have a second $\mathbb{C}^*$ action, also fixed point free, defining a principal bundle $\mathcal{N} \rightarrow P(\mathcal{N})$. Combining the projections

$$\mathcal{S}^2 \rightarrow \mathcal{N} \rightarrow P(\mathcal{N})$$

evidently corresponds to combining the $\mathbb{C}^*$ actions to give a $\mathbb{C}^* \times \mathbb{C}^*$ action on $\mathcal{S} \times \mathcal{S}$. Indeed, since $n = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}$, $tn = t\boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}$, we can take the combined action as

$$((t, \lambda), (\boldsymbol{z}, \boldsymbol{w})) \mapsto (t \lambda \boldsymbol{z}, \lambda^{-1} \boldsymbol{w}).$$

But, writing $t\lambda = \sigma$ and $\lambda^{-1} = \tau$ (so $t = \sigma \lambda^{-1}$, $\lambda = \tau^{-1}$) this latter action is equivalent to the $\mathbb{C}^* \times \mathbb{C}^*$ action given by

$$((\sigma, \tau), (\boldsymbol{z}, \boldsymbol{w})) \mapsto (\sigma \boldsymbol{z}, \tau \boldsymbol{w}).$$

Hence the base $P(\mathscr{N})$ is the set of orbits for this last action, namely

$$P(\mathscr{N}) = (\mathbb{C}^* \times \mathbb{C}^*)\backslash(\mathscr{S} \times \mathscr{S}) \simeq (\mathbb{C}^*\backslash\mathscr{S}) \times (\mathbb{C}^*\backslash\mathscr{S}) \simeq P(\mathbb{C}) \times P(\mathbb{C}) \simeq S^2 \times S^2.$$

Here $P(\mathbb{C})$, usually denoted $P_1(\mathbb{C})$, is the one-dimensional complex projective space, or Riemann sphere S^2. Hence we have

Proposition 5.2. *The composite projection*

$$\mathscr{S}^2 \to \mathscr{N} \to P(\mathscr{N})$$

of the two principal $\mathbb{C}^*$ bundles above gives a principal $\mathbb{C}^* \times \mathbb{C}^*$ bundle

$$\mathscr{S} \times \mathscr{S} \to P(\mathbb{C}) \times P(\mathbb{C}) = P(\mathscr{N})$$

with action defined componentwise.

Remark 5.3. A closely related composite projection is $\mathscr{S}^2 \to P(\mathscr{S}^2) \to P(\mathbb{C})^2$, which may be obtained by combining the actions $(t, (\boldsymbol{z}, \boldsymbol{w})) \mapsto (t\boldsymbol{z}, t\boldsymbol{w})$ and $(\lambda, (\boldsymbol{z}, \boldsymbol{w})) \mapsto (\lambda \boldsymbol{z}, \boldsymbol{w})$. This may well also be useful.

We can now use these results to give a concrete realization of $\mathbb{C}A$ by means of homogeneous functions on $\mathscr{S} \times \mathscr{S}$.

Proposition 5.4. *Let $C^\infty_{1,1}(\mathscr{S}^2, \mathbb{C})$ be the set of functions $\psi : \mathscr{S}^2 \to \mathbb{C}$ satisfying the homogeneity condition*

$$\psi(\sigma \boldsymbol{z}, \tau \boldsymbol{w}) = \sigma\tau\psi(\boldsymbol{z}, \boldsymbol{w}), \text{ all } (\sigma, \tau) \in \mathbb{C}^* \times \mathbb{C}^*.$$

Then we have the following identifications:

$$\begin{aligned} \mathbb{C}A(\mathscr{N}) &= C^\infty_1(\mathscr{N}, \mathbb{C}) = C^\infty(P(\mathscr{N}), \mathbb{C}) \\ &\simeq C^\infty_{1,1}(\mathscr{S}^2, \mathbb{C}) \simeq C^\infty(P(\mathbb{C})^2, \mathbb{C}). \end{aligned}$$

Proof. The identification of the first three spaces is either by definition, or has been given in §4. Given $\psi \in C^\infty_{1,1}(\mathscr{S}^2, \mathbb{C})$, putting $\tau = \sigma^{-1}$ in the homogeneity condition gives, for all $\sigma \in \mathbb{C}^*$,

$$\psi(\sigma \boldsymbol{z}, \sigma^{-1} \boldsymbol{w}) = \psi(\boldsymbol{z}, \boldsymbol{w}).$$

But this is precisely the condition that ψ is constant on each $\mathbb{C}^*$ orbit of Proposition 5.1, so

$$\psi(\boldsymbol{z}, \boldsymbol{w}) = f(\boldsymbol{z}^{\mathrm{T}} \boldsymbol{w})$$

for some C^∞ function f. Putting $\tau = 1$ in the homogeneity condition, and writing $n = \boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}$, gives

$$f(\sigma n) = f(\sigma \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}) = \psi(\sigma \boldsymbol{z}, \boldsymbol{w}) = \sigma\psi(\boldsymbol{z}, \boldsymbol{w}) = \sigma f(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}) = \sigma f(n).$$

Hence $f \in C^\infty_1(\mathscr{N}, \mathbb{C})$. Similarly, each $f \in C^\infty_1(\mathscr{N}, \mathbb{C})$ defines an element of $C^\infty_{1,1}(\mathscr{S}^2, \mathbb{C})$, and this correspondence is easily seen to be a bijection. The final identification follows from the fact that $P(\mathscr{N}) \simeq P(\mathbb{C})^2$. This completes the proof.

The group action of G^2 on $\mathscr{N}$ needed to specify the semi-direct product $\mathbb{C}B(\mathscr{N})_c$ is the restriction to $\mathscr{N}$ of the G^2 action of §4 on $M(2, \mathbb{C})$. Since the points $n \in \mathscr{N}$ are of the form $n = \boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}$, the required action $\mathscr{N} \times G^2 \to \mathscr{N}$ is given by

$$(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}, (g, h)) \mapsto g^{\mathrm{T}}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}h = (\boldsymbol{z}g)^{\mathrm{T}}(\boldsymbol{w}h).$$

That is, it is induced by the action $\mathscr{S}^2 \times G^2 \to \mathscr{S}$ with

$$((\boldsymbol{z}, \boldsymbol{w}), (g, h)) \mapsto (\boldsymbol{z}g, \boldsymbol{w}h).$$

These results may be summarized in

Theorem 5.5. *The group $\mathbb{C}B(\mathcal{N})_c$ may be realized in the following way:*

$$\mathbb{C}B(\mathcal{N})_c \simeq C^{\infty}_{1,1}(\mathcal{S}^2, \mathbb{C}) \circledS_T G^2,$$

where $C^{\infty}_{1,1}(\mathcal{S}^2, \mathbb{C})$ is the set of C^{∞} functions $\psi : \mathcal{S}^2 \to \mathbb{C}$ satisfying, for all $(\sigma, \tau) \in \mathbb{C}^* \times \mathbb{C}^*$

$$\psi(\sigma \boldsymbol{z}, \tau \boldsymbol{w}) = \sigma\tau\psi(\boldsymbol{z}, \boldsymbol{w})$$

and the representation T of G^2 is given by

$$(T(g,h)\,\psi)\,(\boldsymbol{z}, \boldsymbol{w}) = \psi(\boldsymbol{z}g, \boldsymbol{w}h).$$

(*b*) *Unconstrained supertranslations*

In considering representation theory, it is convenient to work with the realization of $\mathbb{C}A(\mathcal{N})$ in terms of unconstrained functions, that is, with the realization as $C^{\infty}(P(\mathcal{N}), \mathbb{C}) \simeq C^{\infty}(P(\mathbb{C})^2, \mathbb{C})$. The (small) price to be paid for 'free' functions is that the coordinates become local, but even this can be overcome by using four charts rather than one.

So I now introduce convenient local coordinates into $\mathcal{S}$. These are chosen to be compatible with the coordinates used for B in McCarthy (1973*a*, 1975) and elsewhere, and to make the action of the maximal compact subgroup $SU(2)^2$ of G^2 as simple as possible.

Given $\boldsymbol{z} \in \mathcal{S}$, assume that $z_2 \neq 0$ in $\boldsymbol{z} = (z_1, z_2)$. Define a 'length' $r(\boldsymbol{z})$ of $\boldsymbol{z}$ and a 'phase' $\theta(\boldsymbol{z})$ of z_2 by

$$r(\boldsymbol{z}) = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}, \; \theta(\boldsymbol{z}) = z_2/|z_2|.$$

Further define

$$z = z_1/z_2, \quad \rho = \rho(z) = r/|z_2| = (|z|^2+1)^{\frac{1}{2}}.$$

Then $\boldsymbol{z}$ can be written as a complex multiple of a vector $\boldsymbol{e} = \boldsymbol{e}(z)$ which has 'unit length', $r(\boldsymbol{e}(z)) = 1$, and second component real. Indeed

$$\boldsymbol{z} = (z_1, z_2) = r(z_1/r, z_2/r) = r\theta(z/\rho(z), 1/\rho(z))$$

and so, writing the vector in the final brackets as $\boldsymbol{e}(z)$, we have

$$\boldsymbol{z} = r(\boldsymbol{z})\,\theta(\boldsymbol{z})\,\boldsymbol{e}(z).$$

Similarly, writing $w = w_1/w_2$, we have

$$\boldsymbol{w} = r(\boldsymbol{w})\,\theta(\boldsymbol{w})\,\boldsymbol{e}(w).$$

Evidently (z, w) are local 'projective' coordinates for $P(\mathbb{C}) \times P(\mathbb{C})$.

Now let $\psi \in C^{\infty}_{1,1}(\mathcal{S}^2, \mathbb{C})$. Then, using the homogeneity condition, we have

$$\begin{aligned}\psi(\boldsymbol{z}, \boldsymbol{w}) &= \psi(r(\boldsymbol{z})\,\theta(\boldsymbol{z})\,\boldsymbol{e}(z), r(\boldsymbol{w})\,\theta(\boldsymbol{w})\,\boldsymbol{e}(w)) \\ &= r(\boldsymbol{z})\,\theta(\boldsymbol{z})\,r(\boldsymbol{w})\,\theta(\boldsymbol{w})\,\psi(\boldsymbol{e}(z), \boldsymbol{e}(w)).\end{aligned}$$

Writing $\psi(\boldsymbol{e}(z), \boldsymbol{e}(w)) = \alpha(z, w)$, this expresses (locally) every $\psi \in C^{\infty}_{1,1}(\mathcal{S}^2, \mathbb{C})$ in terms of a function $\alpha \in C^{\infty}(P(\mathbb{C})^2, \mathbb{C})$ by

$$\psi(\boldsymbol{z}, \boldsymbol{w}) = r(\boldsymbol{z})\,\theta(\boldsymbol{z})\,r(\boldsymbol{w})\,\theta(\boldsymbol{w})\,\alpha(z, w).$$

To find the expression for the $T(g,h)$ operators in terms of the αs note first that if $g \in G$ is

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the components z_1, z_2 of $\mathbf{z}$ transform linearly, so the ratio $z = z_1/z_2$ transforms fraction linearly. Writing zg for the transformed ratio,

$$zg = \frac{(\mathbf{z}g)_1}{(\mathbf{z}g)_2} = \frac{z_1 a + z_2 c}{z_1 b + z_2 d} = \frac{za+c}{zb+d}.$$

Hence we have

$$\psi(\mathbf{z}g, \mathbf{w}h) = r(\mathbf{z}g)\,\theta(\mathbf{z}g)\,r(\mathbf{w}h)\,\theta(\mathbf{w}h)\,\alpha(zg, wh).$$

The action T of G^2 on the ψs in Theorem 5.5 this gives an action, also denoted T, of G^2 on the αs, defined by

$$(T(g,h)\,\psi)\,(\mathbf{z},\mathbf{w}) = r(\mathbf{z})\,\theta(\mathbf{z})\,r(\mathbf{w})\,\theta(\mathbf{w})\,(T(g,h)\,\alpha)\,(z,w).$$

The last two equations give

$$(T(g,h)\,\alpha)\,(z,w) = k_g(z)\,\eta_g(z)\,k_h(w)\,\eta_h(w)\,\alpha(zg,wh),$$

where the factors on the right are defined by

$$k_g(z) = \frac{r(\mathbf{z}g)}{r(\mathbf{z})} = \frac{|(\mathbf{z}g)_2|\,\rho(zg)}{|z_2|\,\rho(z)} = \left\{\frac{|zb+d|^2+|za+c|^2}{|z|^2+1}\right\}^{\frac{1}{2}},$$

$$\eta_g(z) = \frac{\theta(\mathbf{z}g)}{\theta(\mathbf{z})} = \frac{z_1 b + z_2 d}{|z_1 b + z_2 d|}\cdot\frac{|z_2|}{z_2} = \frac{zb+d}{|zb+d|},$$

with similar formulae for $k_h(w)$ and $\eta_h(w)$. Note that the last three formulae are expressed entirely in terms of local coordinates $(z,w)\in P(\mathbb{C})\times P(\mathbb{C})$. Strictly speaking, three more local charts are needed to cover all of $P(\mathbb{C})\times P(\mathbb{C})$ (cf. McCarthy 1973*a*, 1975), related to the above one via (z, w^{-1}), (z^{-1}, w) and (z^{-1}, w^{-1}), but the single one used above will be sufficient here. Summarizing, we have

Theorem 5.6. *The group $\mathbb{C}B(\mathcal{N})_c$ can be realized as*

$$\mathbb{C}B(\mathcal{N})_c = C^\infty(P(\mathbb{C})^2, \mathbb{C}) \circledS_T G^2$$

with semi-direct specified by

$$(T(g,h)\,\alpha)\,(z,w) = k_g(z)\,\eta_g(z)\,k_h(w)\,\eta_h(w)\,\alpha(zg,wh).$$

The original BMS group B_c is, of course, a 'real form' of $\mathbb{C}B(\mathcal{N})_c$; it is instructive to see how. Consider the following subsets:

$$\{(\mathbf{z},\mathbf{w})\in\mathcal{S}\times\mathcal{S}\,|\,\mathbf{w}=\bar{\mathbf{z}}\}\simeq\mathcal{S},$$
$$\{(g,h)\in G\times G\,|\,h=\bar{g}\}\simeq G,$$
$$\{(\sigma,\tau)\in\mathbb{C}^*\times\mathbb{C}^*\,|\,\tau=\bar{\sigma}\}\simeq\mathbb{C}^*,$$

where bar means complex conjugation. Replace $C^\infty_{1,1}(\mathcal{S}^2,\mathbb{C})$ by the space $C^\infty_{1,\bar{1}}(\mathcal{S},\mathbb{R})$ of all C^∞ functions $\psi:\mathcal{S}\to\mathbb{R}$, with $\mathcal{S}$ the above subset of $\mathcal{S}^2$, with homogeneity condition

$$\psi(\alpha\mathbf{z},\bar{\alpha}\bar{\mathbf{z}}) = \alpha\bar{\alpha}\psi(\mathbf{z},\bar{\mathbf{z}}).$$

Replace the group action by

$$(T(g)\,\psi)\,(\mathbf{z},\bar{\mathbf{z}}) = \psi(\mathbf{z}g,\bar{\mathbf{z}}\bar{g}).$$

Then $C^\infty_{1,\bar{1}}(\mathcal{S},\mathbb{R})\simeq C^\infty(P(\mathbb{C}),\mathbb{R})\simeq C^\infty(S^2,\mathbb{R})$ and we have

$$B_c = C^\infty(S^2,\mathbb{R})\circledS_T G,$$

where
$$(T(g)\,\alpha)\,(z) = k_g^2(z)\,\alpha(zg).$$

This gives the realization of B_c used earlier (McCarthy 1973*a*, 1975), and also (in the earlier notation) $K_g(z) = k_g^2(z)$.

6. Representation theory

Irreducible unitary representations (IRS) of B_c were treated in detail in earlier papers, several of which are cited in McCarthy (1975). Much of the general theory carries over to all of the 42 groups treated in §4. For this reason, I confine attention here to the salient points only. Let

$$\mathscr{B} = A \circledS_T L$$

be any one of these (covering) groups, either real or complex, where A is realized by free functions, and L is the appropriate covering group $L(p,q)_c$ or $\mathbb{C}L_c$. Thus $A = C^\infty(Q, k)$ where Q is either $P_+(K)$ or $P(\mathscr{K})$ and k is, respectively, either $\mathbb{R}$ or $\mathbb{C}$. Thus the unconstrained models are here being used for A.

The subgroup A may reasonably be topologized as a (pre) Hilbert space or as a nuclear space by using a natural measure on Q, or, respectively, the C^∞ manifold structure of Q. In the product topology of $A \times L$, $\mathscr{B}$ then becomes a topological group. The structure of $\mathscr{B}$ is specified by that of the two factors, and by the interaction between them, which is given by the action T of L on A. Let A' be the set of continuous linear functionals on A. The action T of L on A determines a dual action T' of L on A' by setting, for each $l \in L$,

$$(T'(l)\,\phi, \alpha) = (\phi, T(l^{-1})\,\alpha),$$

where $\phi \in A'$, $\alpha \in A$ and (ϕ, α) is the value of the linear functional ϕ on $\alpha \in A$. It is this dual action T' on A' which determines the structure of the IRS of $\mathscr{B}$. In fact, one can show that, in the nuclear topology, every IR of $\mathscr{B}$ arises from a measure on A' which is quasi-invariant and ergodic for this action; for details, see McCarthy (1975).

As in earlier work, attention is here confined to measures on A' which are concentrated on single orbits of the L-action T'. (The remaining ergodic measures, not concentrated on single orbits, are called *strictly ergodic*.) The former measures correspond to subsets of A' which are 'indecomposable' in the set theoretic sense; they give rise to IRS of $\mathscr{B}$ which are induced (in a sense generalizing Mackey's (see Mackey 1968)). Now every orbit $O \subset A'$ of L in A' can be identified with a coset space of L. Indeed, choosing a base point $\phi_0 \in O$ we have

$$O \simeq L/L_0,$$

where L_0 is the stabilizer of $\phi_0 \in O$. In the physics literature, L_0 is called the 'little group' of the point $\phi_0 \in O$.

Let W be an irreducible unitary representation of L_0 in a Hilbert space $\mathscr{H}_0$. The coset space O has a unique class of quasi-invariant measures for the L-action; let ν be one of these. Let $\mathscr{H} = L^2(O, \nu, \mathscr{H}_0)$ be the Hilbert space of functions $\psi : O \rightarrow \mathscr{H}_0$ which are square integrable with respect to ν. Then, from this data, an IR of $\mathscr{B}$ in $\mathscr{H}$ may be given explicitly; it is induced from the IR of $A \circledS L_0$ given by

$$V(\alpha, l) = \exp\,[\mathrm{i}(\phi_0, \alpha)]\,W(l)$$

in $\mathscr{H}_0$. Choosing a different base point for the same orbit (hence a conjugate stabilizer L_0), an IR equivalent to W, and a different allowed ν we get an equivalent IR of $\mathscr{B}$.

In fact, the IRS of $\mathscr{B}$ arising from measures concentrated on orbits all arise in this way, and are determined, up to equivalence, by (1) an orbit $O \subset A'$, (2) a class of equivalent IRS of any little group L_0 of any base point $\phi_0 \in O$.

(It may also be mentioned that the problem of determining equivalence classes of IRS arising from strictly ergodic actions is practically hopeless even for locally compact groups, and certainly for $\mathscr{B}$). Explicit formulae for the induced IRS of $\mathscr{B}$ are similar to those given in McCarthy (1975).

While topologies of the nuclear type for $\mathscr{B}$ have been used to give this general theory, the induced IRS can also be constructed for topologies of the Hilbert type. It has been argued elsewhere (Crampin & McCarthy 1974) that the resulting IRS are more 'physical' in the Hilbert topology for $\mathscr{B}$. Indeed, they appear to describe quantum mechanical systems in asymptotically flat space-times, whereas the nuclear IRS seem, rather, to describe 'scattering states' for gravitational systems. Thus, the Hilbert topology IRS of $\mathscr{B}$ seem closely tied to the smooth, bound gravitational source space-times first considered by Bondi *et al.* (1962) and Sachs (1962*a*). By contrast, the nuclear topology IRS appear to allow unbound sources, possibly with infinite energy, and also distributional metric solutions of Einstein's equations (see McCarthy 1978). It is not known how to define asymptotic symmetry groups in such settings (where the space-times need not even be asymptotically flat). Here, I concentrate on the Hilbert topologies for the groups $\mathscr{B}$.

In principle, the induced IRS of any one of the groups $\mathscr{B}$ can be computed by following a programme similar to the one used in earlier papers for B itself. Here, I investigate the IRS of the complexification $\mathbb{C}B(\mathscr{N})_c$ of B. The structure of this complexification was described in detail in §5.

It is interesting to note that a completely different group theoretic approach to quantum gravity has been developed by Isham (1978, 1984). This fascinating work also arrives, independently and from another point of view, at distributional metrics. In Isham's work, however, these metrics occupy the centre of the stage.

7. Compact little groups for $\mathbb{C}B(\mathscr{N})_c$

The reason for the prominence of the A, D, E series in the theory of IRS of B is that the little groups of B turn out to be compact (McCarthy 1973), for B in the Hilbert topology. In fact, the little groups are precisely the closed subgroups of $SU(2)$, the maximal compact subgroup of $SL(2, \mathbb{C})$, which project onto compact subgroups of $SO(3)$ under the covering map $\gamma: SU(2) \to SO(3, \mathbb{R})$. The A, D, E related groups are simply the finite groups of this type. There is no reason at all to suspect that a similar result should hold for $\mathbb{C}B(\mathscr{N})_c$. In this section, I show that, in fact, all little groups for $\mathbb{C}B(\mathscr{N})_c$ are actually compact. This is especially significant for quantum gravity, because of the relationship to other approaches.

For ease of notation, I write $\mathbb{C}B$ for $\mathbb{C}B(\mathscr{N})_c$ in the rest of this paper. Also, I sometimes write $\mathscr{P}$ for $S^2 \times S^2$ and $\mathscr{G}$ for $G \times G$. So, by Theorem 5.6, $\mathbb{C}B$ has structure

$$\mathbb{C}B = C^\infty(\mathscr{P}, \mathbb{C}) \circledS_T \mathscr{G}.$$

In analogy to B, it is natural to choose a measure λ on $\mathscr{P} = S^2 \times S^2$ which is invariant under the maximal compact subgroup $SU(2) \times SU(2)$ of $\mathscr{G} = G \times G$. It will be convenient to use, as coordinates for $\mathscr{P} = S^2 \times S^2$, the six components of a pair $(\boldsymbol{m}, \boldsymbol{n})$ of unit length vectors in $\mathbb{R}^3$

$$\mathscr{P} = S^2 \times S^2 = \{(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, |\boldsymbol{m}| = 1, |\boldsymbol{n}| = 1\}.$$

These coordinates are related to the 'projective' coordinates (z, w) for $\mathscr{P} = S^2 \times S^2$ by stereographic projection. The advantage is that the $(\boldsymbol{m}, \boldsymbol{n})$ coordinates are globally defined.

The required measure on $\mathscr{P}$ is given by the 4-form

$$\mathrm{d}\lambda(\boldsymbol{m}, \boldsymbol{n}) = \mathrm{d}\mu(\boldsymbol{m}) \wedge \mathrm{d}\mu(\boldsymbol{n}),$$

where each factor on the right is the usual invariant (normalized) surface element for S^2. The explicit form is, for each factor,

$$\mathrm{d}\mu(\boldsymbol{m}) = (\mathrm{d}m_1 \wedge \mathrm{d}m_2)/4\pi\,|m_3|,$$

where m_1, m_2, m_3 are components of $\boldsymbol{m} \in \mathbb{R}^3$. This formula is only valid for $m_3 \neq 0$, but any cyclic permutation of the indices in the components gives another valid expression for $\mathrm{d}\mu(\boldsymbol{m})$. In projective coordinates $z \in P(\mathbb{C})$, the corresponding expression is

$$\mathrm{d}\mu(z) = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2}.$$

Letting $\boldsymbol{m}g$ be the transform of $\boldsymbol{m} \in S^2 = P(\mathbb{C})$ by $g \in SL(2, \mathbb{C})$ (so that $z \mapsto zg$ corresponds to $\boldsymbol{m} \mapsto \boldsymbol{m}g$), a simple calculation gives

$$\mathrm{d}\mu(\boldsymbol{m})/\mathrm{d}\mu(\boldsymbol{m}g) = k_g^4(z) = k_g^4(\boldsymbol{m}).$$

Here $k_g(\boldsymbol{m})$ denotes $k_g(z)$, where $\boldsymbol{m}$ corresponds to z. While the expressions given for $\mathrm{d}\mu(\boldsymbol{m})$ are local, $k_g(\boldsymbol{m})$ is defined globally.

A pre-Hilbert space structure can now be given to $C^\infty(\mathscr{P}, \mathbb{C})$ by defining a scalar product

$$\langle \alpha, \beta \rangle = \int_{\mathscr{P}} \bar{\alpha}(\boldsymbol{m}, \boldsymbol{n})\, \beta(\boldsymbol{m}, \boldsymbol{n})\, \mathrm{d}\lambda(\boldsymbol{m}, \boldsymbol{n})$$

$$= \int_{\mathscr{P}} \bar{\alpha}(z, w)\, \beta(z, w)\, \mathrm{d}\mu(z) \wedge \mathrm{d}\mu(w),$$

where $\alpha, \beta \in C^\infty(\mathscr{P}, \mathbb{C})$ and the complex conjugate is defined pointwise. As for B, it is convenient to complete the space with respect to the norm defined by the scalar product. In the resulting Hilbert space, functions are identified whenever they differ, at most, on a set of measure zero. Thus our Hilbert space is

$$L^2 = L^2(\mathscr{P}, \lambda, \mathbb{C}).$$

Note that $\alpha \in L^2$ is non-vanishing if and only if $\|\alpha\| > 0$, where $\|\alpha\|$ is the L^2 norm. So we now deal with the (complete) group

$$\mathbb{C}B = L^2(\mathscr{P}, \lambda, \mathbb{C}) \circledS_T \mathscr{G}$$

with the action T of $\mathscr{G} = G \times G$ on the new $\mathbb{C}A = L^2(\mathscr{P}, \lambda, \mathbb{C})$ given by the same formula as before:

$$(T(g,h)\,\alpha)\,(z,w) = k_g(z)\,\eta_g(z)\,k_h(w)\,\eta_h(w)\,\alpha(zg, wh).$$

It is well known that the topological dual of a Hilbert space can be identified with the Hilbert space itself, so that we now have $\mathbb{C}A' \simeq \mathbb{C}A$. In fact, given a continuous linear functional $\phi \in \mathbb{C}A'$, we can write, for $\alpha \in \mathbb{C}A$

$$(\phi, \alpha) = \langle \phi, \alpha \rangle,$$

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where the function $\phi \in \mathbb{C}A$ on the right is uniquely determined by (and denoted by the same symbol as) the linear functional $\phi \in \mathbb{C}A'$ on the left.

The representation theory of $\mathbb{C}B$ is governed by the dual action T' of $\mathscr{G} = G \times G$ on $\mathbb{C}A' \simeq \mathbb{C}A$, defined as in §6:

$$\langle T'(g,h)\,\phi, \alpha\rangle = \langle \phi, T(g^{-1}, h^{-1})\,\alpha\rangle.$$

A short calculation, involving a simple change of variables, gives

$$\langle T'(g,h)\,\phi, \alpha\rangle = \int_{\mathscr{P}} k_g^{-5}(z)\,\bar{\eta}_g(z)\,k_h^{-5}(w)\,\bar{\eta}_h(w)\,\bar{\phi}(zg, wh)\,\alpha(z,w)\,\mathrm{d}\lambda(z,w).$$

Since this holds for all $\phi \in \mathbb{C}A$,

$$(T'(g,h)\,\phi)\,(z,w) = k_g^{-5}(z)\,\eta_g(z)\,k_h^{-5}(w)\,\eta_h(w)\,\phi(zg, wh).$$

It is convenient to have the corresponding transformation law for the modulus $|\phi(z,w)|$:

$$|(T'(g,h)\,\phi)\,(\boldsymbol{m},\boldsymbol{n})| = k_g^{-5}(\boldsymbol{m})\,k_h^{-5}(\boldsymbol{n})\,|\phi(\boldsymbol{m}g, \boldsymbol{n}h)|.$$

Now, this action T' of $\mathscr{G}$ on A', given explicitly above, is, like the action T of $\mathscr{G}$ on A, continuous. The 'little group' L_ϕ of any $\phi \in \mathbb{C}A'$ is the stabilizer

$$L_\phi = \{(g,h) \in G \times G \mid T'(g,h)\,\phi = \phi\}.$$

By continuity, $L_\phi \subset \mathscr{G}$ is a closed subgroup. Clearly, if $\phi = 0$, $L_\phi = \mathscr{G}$ and the orbit of $\phi = 0$ is just the origin. The resulting IRS of $\mathbb{C}B$ are trivial in the subgroup A, and are really only IRS of $\mathscr{G}$. (In fact, they are just the IRS of $\mathscr{G}$ with which the inducing starts.) These representations are unphysical, since they describe quantum systems of 'zero supermomentum'; they are ignored here, as in earlier work for B. Henceforth, then, I assume that $\phi \neq 0$.

I now prove the following:

Theorem 7.1. *Let $\phi \in \mathbb{C}A' - 0$ have stabilizer $L_\phi \subset \mathscr{G}$. Then this 'little group' L_ϕ is compact. That is, every little group of $\mathbb{C}B$ is compact.*

Proof. Assume, contrary to the theorem, that L_ϕ is non-compact for some $\phi \neq 0$. Then $L_\phi \subset \mathscr{G}$ must be unbounded with respect to a standard metric for $\mathscr{G}$. Indeed, L_ϕ is closed, so if it were bounded, it would be compact, contrary to our starting assumption. Since $\phi \in \mathbb{C}A'$, ϕ is square integrable, and since $(1-|\phi|)^2 \geqslant 0$ gives $|\phi| \leqslant \frac{1}{2}(1+|\phi|^2)$, $|\phi|$ is integrable. By definition, $T'(g,h)\,\phi = \phi$ for all $(g,h) \in L_\phi$. Hence, for all $(g,h) \in L_\phi$,

$$\int_{\mathscr{P}} |(T'(g,h)\,\phi)\,(\boldsymbol{m},\boldsymbol{n})|\,\mathrm{d}\lambda(\boldsymbol{m},\boldsymbol{n}) = \int_{\mathscr{P}} |\phi(\boldsymbol{m},\boldsymbol{n})|\,\mathrm{d}\lambda(\boldsymbol{m},\boldsymbol{n}).$$

Substituting for $T'(g,h)\,\phi$ from above gives, after simple manipulations and changing variables,

$$\int_{\mathscr{P}} k_{g^{-1}}(\boldsymbol{m})\,k_{h^{-1}}(\boldsymbol{n})\,|\phi(\boldsymbol{m},\boldsymbol{n})|\,\mathrm{d}\lambda(\boldsymbol{m},\boldsymbol{n}) = \int_{\mathscr{P}} |\phi(\boldsymbol{m},\boldsymbol{n})|\,\mathrm{d}\lambda(\boldsymbol{m},\boldsymbol{n})$$

for all $(g,h) \in L_\phi$. Since L_ϕ is a group, the same equation holds with (g^{-1}, h^{-1}) replaced by (g,h).

Write $p = (\boldsymbol{m},\boldsymbol{n})$, $\gamma = (g,h)$ and write $C(\gamma,p) = k_g(\boldsymbol{m})\,k_h(\boldsymbol{n})$. Then the previous equation gives

$$\int_{\mathscr{P}} (C(\gamma,p)-1)\,|\phi(p)|\,\mathrm{d}\lambda(p) = 0$$

for all $\gamma \in L_\phi$. It will be shown in the Appendix that the unbounded property of L_ϕ implies that there is an infinite sequence $(\gamma_n) \subset L_\phi$, $n = 1, 2, 3, \ldots$, with the following properties. For any given $M > 0$, fixed throughout the discussion, define, for each n, the subsets P_n of $\mathscr{P}$ (which depend on M) by

$$P_n = \{p \in \mathscr{P} \mid C(\gamma_n, p) > M\},$$

with complements

$$P'_n = \{p \in \mathscr{P} \mid C(\gamma_n, p) \leqslant M\}.$$

Then the $\mathscr{P}$-measure $\lambda(P_n)$ of P_n satisfies

$$\lambda(P_n) \to 1 \quad \text{as} \quad n \to \infty.$$

Since the measure is normalized and $\mathscr{P} = P_n \cup P'_n$ is a disjoint union, $\lambda(P_n) + \lambda(P'_n) = 1$ and so $\lambda(P_n) \to 1$ is equivalent to

$$\lambda(P'_n) \to 0 \quad \text{as} \quad n \to \infty.$$

The above integral over $\mathscr{P}$ gives, in obvious shorthand notation,

$$\int_{P_n} (C(\gamma_n) - 1)\, |\phi| \, \mathrm{d}\lambda = \int_{P'_n} (1 - C(\gamma_n))\, |\phi| \, \mathrm{d}\lambda.$$

Adding the same term to each side,

$$\int_{P_n} (C(\gamma_n) - 1)\, |\phi| \, \mathrm{d}\lambda + \int_{P'_n} |\phi| \, \mathrm{d}\lambda = \int_{P'_n} (2 - C(\gamma_n))\, |\phi| \, \mathrm{d}\lambda.$$

Now choose $M = 2$, giving, for all sufficiently large n,

$$\int_{\mathscr{P}} |\phi| \, \mathrm{d}\lambda \leqslant \int_{P_n} (C(\gamma_n) - 1)\, |\phi| \, \mathrm{d}\lambda + \int_{P'_n} |\phi| \, \mathrm{d}\lambda$$

$$= \int_{P'_n} (2 - C(\gamma_n))\, |\phi| \, \mathrm{d}\lambda \leqslant \int_{P'_n} 2|\phi| \, \mathrm{d}\lambda.$$

Hence, for all sufficiently large n,

$$\int_{\mathscr{P}} |\phi| \, \mathrm{d}\lambda \leqslant 2 \int_{P'_n} |\phi| \, \mathrm{d}\lambda.$$

Letting $\chi(P'_n)$ denote the characteristic function of P'_n, the Cauchy–Schwartz inequality gives

$$\left| \int_{P'_n} |\phi| \, \mathrm{d}\lambda \right|^2 = \left| \int_{\mathscr{P}} \chi(P'_n)\, |\phi| \, \mathrm{d}\lambda \right|^2$$

$$\leqslant \left(\int_{\mathscr{P}} |\chi(P'_n)|^2 \, \mathrm{d}\lambda \right) \left(\int_{\mathscr{P}} |\phi|^2 \, \mathrm{d}\lambda \right)$$

$$= \lambda(P'_n)\, \|\phi\|^2.$$

Hence we have, for all sufficiently large n,

$$\int_{\mathscr{P}} |\phi| \, \mathrm{d}\lambda \leqslant 2 (\lambda(P'_n))^{\frac{1}{2}} \|\phi\|.$$

But $\lambda(P'_n) \to 0$ as $n \to \infty$, so the last inequality is only possible if $\int_{\mathscr{P}} |\phi| \, d\lambda = 0$. But then ϕ must vanish almost everywhere, so $\phi = 0$ as an element of $\mathbb{C}A' \simeq \mathbb{C}A$. This contradicts $\phi \in \mathbb{C}A' - 0$. So L_ϕ cannot be non-compact and the theorem is proved.

The representation T of G^2 defining $\mathbb{C}B$ actually acts via the covering map $\gamma: G^2 \to \mathbb{C}L$, so every little group $L_\phi \subset \mathscr{G}$ has a well-defined projection $\gamma: G^2 \to \mathbb{C}L$. The theorem shows that, up to conjugacy, every $L_\phi \subset \mathscr{G}$ is also a closed subgroup of the maximal compact subgroup $SU(2) \times SU(2)$ of $\mathscr{G}$. Let π_i, $i = 1, 2$, be the projections onto the factors of $SU(2) \times SU(2)$. Since these are closed maps, the $\pi_i(L_\phi)$ are closed subgroups of $SU(2)$. Hence we have

Corollary 7.2. *Let L_ϕ be a little group of $\mathbb{C}B$, and let $\gamma: SU(2) \times SU(2) \to SO(4, \mathbb{R})$ be the restriction of the above to the maximal compact subgroup of G^2. Then, up to conjugacy,*
(1) $L_\phi = \gamma^{-1}(C)$,
(2) *L_ϕ is a closed subgroup of $\pi_1^{-1}(G_1) \times \pi_2^{-1}(G_2)$,*
where C is a compact subgroup of $SO(4, \mathbb{R})$, and G_1, G_2 are compact subgroups of $SU(2)$.

In particular, if L_ϕ is finite, we have

Corollary 7.3. *Let L_ϕ be a finite little group of $\mathbb{C}B$. Then*
(1) $L_\phi = \gamma^{-1}(F)$,
(2) *L_ϕ is a subgroup of $F_1 \times F_2$,*
where F is a finite subgroup of $SO(4, \mathbb{R})$, and F_1, F_2 are (independently) cyclic, binary dihedral, binary tetrahedral, binary octahedral or binary icosahedral subgroups of $SU(2)$.

This final result shows that the A, D, E series appears in IRs of in a more subtle way than for B. Indeed, here we get products and subgroups of products of these finite groups. More details will appear elsewhere.

8. Some remarks on *EB*

In §2, I mentioned a close link between complex instantons and complex supertranslations via the good cut equation, and in particular, Hitchin's work (1982) on the good cut equation and the euclidean Eguchi–Hanson (Eguchi & Hanson 1978) metric. This suggests that euclidean instantons are closely tied to representations of $\mathbb{C}B$. The little groups of the latter are subgroups of the homogeneous part $G^2 = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, and since $SL(2, \mathbb{C})$ is non-compact, some of these little groups themselves might have been non-compact. §7 shows that, in fact, they are necessarily compact.

Also, in §2, I mentioned the derivation of EB as the universal symmetry group of asymptotically euclidean space-times. However, since

$$EB_c = C^\infty(S^3, \mathbb{R}) \circledS_T (SU(2) \times SU(2)),$$

the little groups for EB_c are automatically compact, being closed subgroups of the homogeneous part $SU(2)^2$. Concerning possible links between these representations and euclidean instantons, I here note the following.

(i) The boundary conditions for (ALE) euclidean instantons normally require that, as $r \to \infty$, the metric approaches flatness faster than the rate required in §2.

(ii) For (ALE) euclidean instantons, the metric locally approaches flatness, but a neighbourhood of infinity has topology $I \times (S^3/\Gamma)$, where Γ is a finite group of isometries acting freely on S^3 (see, for example, Gibbons & Pope 1979; Hitchin 1979).

In the lorentzian case, in the IRs of B, the 'supermomenta' of the IRs had a key physical significance (Crampin & McCarthy 1974). These supermomenta have the

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same transformation law as 'mass aspects', as mentioned in that paper. Now these 'mass aspects' appear, in the Bondi–Sachs metric, as coefficients of higher powers of (the analogue of) r^{-1} than the first (see the discussion by Bondi (1965)). Also, in induced IRS of B, the supermomenta appear as having precisely the symmetries of the group (typically a finite polygon or polyhedral one) with which the induction starts (McCarthy 1973*b*). It seems likely that analogous results will apply in the case of EB, and will link (i) and (ii) with the corresponding properties of the IRS.

9. The action of $\mathbb{C}B$ on $\mathbb{C}\mathscr{I}^+$

In approaching quantum theory via symmetries,it is appropriate firstly to define clearly the abstract group concerned, and secondly to find the possible group actions (i.e. representations) in (projective) Hilbert spaces. This plan has been followed above; various generalizations of B were constructed abstractly, and the IRS (especially for $\mathbb{C}B$) investigated in Hilbert spaces. However, B first arose as an asymptotic symmetry group on space-times (Bondi *et al.* 1962; Sachs 1962*a*), then as an actual transformation group of the future null infinity $\mathscr{I}^+$ of these space-times (Penrose 1963, 1974). In fact, B is precisely the symmetry group of $\mathscr{I}^+$. It is evidently important to examine this classical action on the space-time boundary $\mathscr{I}^+$, and to find the analogue for $\mathbb{C}B$.

In dealing with the abstract groups $B^{p,q}(K) = A(K) \circledS_T L$ it was convenient, in specifying the action T if L on A, to use the action of L on K from the right. This avoids a plethora of group inverses. However, to agree with notations used elsewhere (e.g. in Penrose & Rindler (1986)) for the classical action $B \times \mathscr{I}^+ \to \mathscr{I}^+$ from the left, it now becomes convenient to use the action of L on K from the left. So, for this section only, I redefine the groups and quantities needed. Let $g \in G$ and write

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad g\mathbf{z} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

$$z = z_1/z_2, \quad gz = (az+b)/(cz+d).$$

Now, with $r(\mathbf{z}) = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$ and $\theta(\mathbf{z}) = z_2/|z_2|$, redefine

$$k_g(z) = \frac{r(g\mathbf{z})}{r(\mathbf{z})} = \left\{ \frac{|az+b|^2 + |cz+d|^2}{|z|^2+1} \right\}^{\frac{1}{2}}$$

$$\eta_g(z) = \theta(g\mathbf{z})/\theta(\mathbf{z}) = (cz+d)/|cz+d|.$$

Also write $K_g(z) = k_g^2(z)$.

With these new definitions, adapted to the left action of G on one can now redefine

$$B_c = C^\infty(P(\mathbb{C}), \mathbb{R}) \circledS_T G,$$

where now $$(T(g)\,\alpha)\,(z) = k^2_{g^{-1}}(z)\,\alpha(g^{-1}z) = K_{g^{-1}}(z)\,\alpha(g^{-1}z).$$

Also redefine $$\mathbb{C}B_c = C^\infty(P(\mathbb{C})^2, \mathbb{C}) \circledS_T G^2,$$

where now the G^2 action T is given by

$$(T(g,h)\,\alpha)\,(z,w) = k_{g^{-1}}(z)\,\eta_{g^{-1}}(z)\,k_{h^{-1}}(w)\,\eta_{h^{-1}}(w)\,\alpha(g^{-1}z, h^{-1}w).$$

It should be clear from this last formula why the inverses were avoided earlier.

$\mathscr{I}^+$ can be identified with the product $\mathbb{R} \times P(\mathbb{C})$, and the classical action (given by many authors, e.g. Penrose & Rindler (1986)) may, after a convenient reformulation,

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be described as follows. Let $[u,z] \in \mathbb{R} \times P(\mathbb{C}) = \mathscr{I}^+$. Then $(\alpha, g) \in B_c$ may be obtained as the product $(\alpha, g) = (\alpha, Id)(0, g)$ (pure Lorentz followed by pure supertranslation). The separate actions are, in order,

$$(0,g)[u,z] = [K_g^{-1}(z)\,u, gz] \in \mathscr{I}^+,$$
$$(\alpha, Id)[u,z] = [u+\alpha(z), z] \in \mathscr{I}^+.$$

Combining the two gives

$$(\alpha, g)[u,z] = [K_g^{-1}(z)\,u + \alpha(gz), gz].$$

This last formula, then, gives the classical action $B \times \mathscr{I}^+ \to \mathscr{I}^+$. Applying a further (β, h) gives

$$(\beta, h)(\alpha, g) = (\beta + T(h)\,\alpha, hg),$$

where $T(h)\,\alpha$ is easily checked to be given by the action T defining B_c in the previous paragraph.

The structure $\mathbb{C}\mathscr{I}^+$ is more complicated than that of $\mathscr{I}^+$; it is no longer a product. But it may still be identified locally with the product $\mathbb{C} \times P(\mathbb{C})^2$. I now give a local formula for the action of $\mathbb{C}B$ on $\mathbb{C}\mathscr{I}^+$, using the local identification

$$[u,z,w] \in \mathbb{C} \times P(\mathbb{C}) \times P(\mathbb{C}) \subset \mathbb{C}\mathscr{I}^+.$$

Let $(\alpha, (g,h)) \in \mathbb{C}B = C^\infty(P(\mathbb{C})^2, \mathbb{C}) \circledS_T G^2$. Then the required classical action is given locally by

$$(\alpha, (g,h))[u,z,w] = [k_g^{-1}(z)\,\eta_g^{-1}(z)\,k_h^{-1}(w)\,\eta_h^{-1}(w)\,u + \alpha(gz, hw), gz, hw].$$

Applying a second group element of $\mathbb{C}B_c$ gives the correct group product. The local action $\mathbb{C}B_c \times \mathbb{C}\mathscr{I}^+ \to \mathbb{C}\mathscr{I}^+$ is the required complexification of $B_c \times \mathscr{I}^+ \to \mathscr{I}^+$. Both actions are transitive, and exhibit B_c, $\mathbb{C}B_c$ respectively as symmetry groups of $\mathscr{I}^+$ and $\mathbb{C}\mathscr{I}^+$.

10. Historical remarks

Since the topics of the BMS group, and also of the early work on gravitational instantons, are relatively old, it seems appropriate here to make some historical remarks. The BMS group was discovered (in restricted form) by Bondi, van der Burg and Metzner (Bondi *et al.* 1962) and (in general form) by Sachs (1962*a*). The first IR was found by Sachs (1962*b*). The semi-direct product structure of B was pointed out by Cantoni (1967*a*) and Foster (1966). The first class of IRS was found by Cantoni (1966, 1967*a*, *b*), who constructed these IRS from IRS of the Poincaré subgroup of P of B.

The particular representation of L on A which specifies the semi-direct product B was pointed out by Geroch & Newman (1971), and in more detail by myself (1972*b*). Using an infinite-dimensional modification of Mackey theory, I constructed the set of all induced IRS in a series of papers, several of which are cited in McCarthy (1975). Here, the polygon and polyhedral group IRS arose, completely naturally, in McCarthy (1973*a*). Some of the papers in this series were written in collaboration with M. Crampin.

Following the analogy of Yang–Mills theory, Eguchi & Freund (1976) found a gravitational instanton, and Hawking (1977) a related 'many Taub–NUT' solution. The latter is a so called 'ALF' solution, asymptotically locally flat in a three-dimensional spatial sense, and periodic in imaginary time. Eguchi & Hanson (1978) found all 1-instanton solutions of the euclidean self-dual Einstein equations, but now

'ALE' (asymptotically locally euclidean) in a four-dimensional sense, with the metric approaching the flat metric of $\mathbb{R}^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by $x \mapsto x, x \mapsto -x$. Gibbons & Hawking (1978) and Gibbons & Pope (1979) observed that certain ALE instantons were controlled by the cyclic symmetry groups Γ of polygons, the metric approaching the flat metric of $\mathbb{R}^4/\Gamma$. Hitchin (1979) gave a twistor-related construction of these solutions, and argued that similar constructions should apply for the dihedral groups and the groups of polyhedra as well. Kronheimer (1986, 1989*a*, *b*) provided a beautiful (but still constrained) description of the moduli space of all gravitational multi-instantons, which now involves the complex linear irreducible representations of the polygon and polyhedral groups.

In the present context, the possible link with IRS of EB would appear to refer to ALE rather than ALF metrics. Indeed, for EB the supertranslations are functions on S^3 whereas, for the ALF boundary conditions, one would expect functions on $S^2 \times S^1$. This seems to apply also to the link with $\mathbb{C}B$. Further investigation of this point would certainly be of interest.

It is a pleasure to thank Sir Hermann Bondi for his encouragement and support of this work. I also thank Malcolm MacCallum, whose study groups in Queen Mary College have refreshed my memory in several topics in general relativity, a research field which I had left for some time to follow other areas of mathematical physics.

I also thank a referee or an earlier version of this paper (McCarthy 1989) for a very helpful (and positive) report. This referee brought the work of Ashteker & Hansen (1978) to my attention, and suggested that their group might be related to my EB, possibly via my complexification $\mathbb{C}EB$. (My earlier paper only discussed B, EB, HB, and their complexifications $\mathbb{C}B = \mathbb{C}HB$, $\mathbb{C}EB$, and also quantum gravity and the A, D, E series.) These remarks were stimulating for my eventual construction of a wide variety of further new groups. The same referee also suggested describing $\mathbb{C}B$ as a symmetry group of $\mathbb{C}\mathscr{I}^+$ and including the distinction between ALE and ALF instantons. I thank another referee of my earlier paper for drawing my attention to the work of Hitchin (1982). Finally, I am grateful to two referees of this paper for their positive reports.

Appendix

In Theorem 7.1, a property of unbounded subgroups L_ϕ of $\mathscr{G}$ was used. Here the proof is given. The standard topology of $G = SL(2, \mathbb{C})$ may be given by a metric derived from the norm

$$|g| = (|a|^2 + |b|^2 + |c|^2 + |d|^2)^{\frac{1}{2}}, \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G.$$

Note that, for all $g_1, g_2 \in G$

$$|g_1 g_2| \leqslant |g_1|\,|g_2|. \tag{A 1}$$

The metric on $\mathscr{G} = G \times G$ can be defined by

$$|\gamma| = (|g|^2 + |h|^2)^{\frac{1}{2}}, \quad \gamma = (g, h) \in G \times G.$$

Since L_ϕ is unbounded, there is a sequence (γ_n) in L_ϕ, $n = 1, 2, 3, \ldots$, such that $|\gamma_n| \to \infty$ as $n \to \infty$. Henceforth, all limits refer to $n \to \infty$. Therefore there are three possibilities for the components g_n and h_n in $\gamma_n = (g_n, h_n)$, namely

(A) $|g_n|$ is unbounded and $|h_n|$ is unbounded,

(B) $|g_n|$ is bounded and $|h_n|$ is unbounded,

(C) $|h_n|$ is bounded and $|g_n|$ is unbounded.

Passing to subsequences if necessary, we may assume, then, that

$$\begin{array}{llll} \text{(A)} & |g_n| \to \infty & \text{and} & |h_n| \to \infty, \\ \text{(B)} & |g_n| < c & \text{and} & |h_n| \to \infty, \\ \text{(C)} & |h_n| < d & \text{and} & |g_n| \to \infty, \end{array}$$

where c and d are positive constants.

The required result depends on the functions $C(\gamma_n, p) = k_{g_n}(\boldsymbol{m})\, k_{h_n}(\boldsymbol{n})$, $\gamma_n = (g_n, h_n)$, $p = (\boldsymbol{m}, \boldsymbol{n})$, which are positive real functions on $S^2 \times S^2$, and is that, given any $M > 0$, the subsets

$$P_n = \{(\boldsymbol{m}, \boldsymbol{n}) \in S^2 \times S^2 | k_{g_n}(\boldsymbol{m})\, k_{h_n}(\boldsymbol{n}) > M\} \tag{A 2}$$

satisfy $\lambda(P_n) \to 1$. Equivalently, the complements P'_n satisfy $\lambda(P'_n) \to 0$.

Consider first the function $K_g : S^2 \to \mathbb{R}^+$ defined by

$$K_g(\boldsymbol{m}) = k_g^2(\boldsymbol{m}). \tag{A 3}$$

It is well known that every $g \in G$ can be written in the form

$$g = uaw, a = a(t) = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}, \tag{A 4}$$

where $u, w \in SU(2)$ and t is real. Since, for all t,

$$a(t) = Ja(-t)\, J^{-1}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SU(2),$$

we can, and henceforth will, assume, replacing u, w by $uJ, J^{-1}w$ if necessary, that $t \geqslant 0$ for all g in the form (A 4). It can easily be shown that $K_g(\boldsymbol{m})$ has the form, for $g = uaw$,

$$K_g(\boldsymbol{m}) = \cosh t + \sinh t(\boldsymbol{v} \cdot \boldsymbol{m}), \tag{A 5}$$

where $\boldsymbol{v} \in S^2 \subset \mathbb{R}^3$ is a unit vector uniquely determined by w. Note that, since the function $\boldsymbol{v} \cdot \boldsymbol{m}$ maps S^2 onto the closed interval $[-1, 1]$, K_g maps S^2 onto the closed interval $[e^{-t}, e^t]$.

Now suppose that (g_n) is a sequence in G with $|g_n| \to \infty$ as $n \to \infty$, and let $g_n = u_n a_n w_n$ be the corresponding decompositions (A 4), with $a_n = a(t_n)$, $t_n \geqslant 0$. The $w_n \in SU(2)$ determine, in the formulae for $K_{g_n}(\boldsymbol{m})$, an infinite sequence of unit vectors in the compact space $S^2 \subset \mathbb{R}^3$. Hence, by passing to a subsequence (g_{n_i}) we may assume that the $\boldsymbol{v}_{n_i}$ converge to a limit unit vector $\boldsymbol{k} \in S^2$. Dropping the extra index for the subsequence, then, $\boldsymbol{v}_n = \boldsymbol{k} + \boldsymbol{\alpha}_n$, where $\boldsymbol{\alpha}_n \to \boldsymbol{0}$ as $n \to \infty$. So $\boldsymbol{v}_n \cdot \boldsymbol{m} = \boldsymbol{k} \cdot \boldsymbol{m} + \beta_n(\boldsymbol{m})$, where $|\beta_n(\boldsymbol{m})| = |\boldsymbol{\alpha}_n \cdot \boldsymbol{m}| \leqslant |\boldsymbol{\alpha}_n| \to 0$ uniformly on S^2 as $n \to \infty$. Since the norm of every $SU(2)$ matrix is $\sqrt{2}$, (A 4) for g_n gives $|g_n| \leqslant 2|a_n|$, so $|a_n| \to \infty$, hence $|a_n|^2 = 2 \cosh t_n \to \infty$ so $t_n \to \infty$ as $n \to \infty$. So (A 5) gives

$$\begin{aligned} K_{g_n}(\boldsymbol{m}) &= \tfrac{1}{2} e^{t_n}(1 + \boldsymbol{k} \cdot \boldsymbol{m} + \beta_n) + \tfrac{1}{2} e^{-t_n}(1 - \boldsymbol{k} \cdot \boldsymbol{m} - \beta_n) \\ &= \tfrac{1}{2} e^{t_n}(1 + \boldsymbol{k} \cdot \boldsymbol{m} + \beta_n) + \gamma_n \end{aligned} \tag{A 6}$$

say, where the second term γ_n on the right also tends to 0 uniformly on S^2, since $e^{-t_n} \to 0$.

Now suppose that some $M > 0$ is given, and let

$$R_n = \{\boldsymbol{m} \in S^2 \,|\, K_{g_n}(\boldsymbol{m}) > M\}. \tag{A 7}$$

Let R'_n be the complement of R_n in S^2. (Note that the image of $K_{g_n}: S^2 \to \mathbb{R}^+$ is the closed interval $[e^{-t_n}, e^{t_n}]$, so both R_n and R'_n are non-empty for sufficiently large n.) Then (A 6) gives, for $\boldsymbol{m} \in R'_n$

$$1 + \boldsymbol{k}\cdot\boldsymbol{m} + \beta_n \leqslant 2\,e^{-t_n}(M - \gamma_n). \tag{A 8}$$

Since $e^{-t_n} \to 0$, the right-hand side δ_n of (A 8) tends to 0, so $\boldsymbol{k}\cdot\boldsymbol{m} \leqslant \epsilon_n - 1$, where $\epsilon_n = \delta_n - \beta_n$ also tends to 0. Also, for all $\boldsymbol{m} \in S^2$, $-1 \leqslant \boldsymbol{k}\cdot\boldsymbol{m}$ so $\boldsymbol{m} \in R'_n$ implies

$$-1 \leqslant \boldsymbol{k}\cdot\boldsymbol{m} \leqslant \epsilon_n - 1. \tag{A 9}$$

In fact, (A 9) also implies $\epsilon_n \geqslant 0$ for all n. The surface area of S^2 defined by the inequality (A 9) is $2\pi\epsilon_n$. So the normalized measure is $(2\pi\epsilon_n)/4\pi = \frac{1}{2}\epsilon_n$. Hence the S^2 measure μ of R'_n satisfies

$$0 \leqslant \mu(R'_n) \leqslant \tfrac{1}{2}\epsilon_n.$$

Since $\epsilon_n \to 0$, $\mu(R'_n) \to 0$ and so $\mu(R_n) \to 1$.

Now suppose that possibility (A) applies to $\gamma_n = (g_n, h_n)$, and define

$$R_n = \{\boldsymbol{m} \in S^2 \,|\, K_{g_n}(\boldsymbol{m}) > M\}, \quad T_n = \{\boldsymbol{n} \in S^2 \,|\, K_{h_n}(\boldsymbol{n}) > M\}.$$

Then, as proved above, $\mu(R_n) \to 1$ and $\mu(T_n) \to 1$. But

$$\begin{aligned} R_n \times T_n &= \{(\boldsymbol{m}, \boldsymbol{n}) \in \mathscr{P} \,|\, k_{g_n}(\boldsymbol{m}) > \sqrt{M}, k_{h_n}(\boldsymbol{n}) > \sqrt{M}\} \\ &\subset \{(\boldsymbol{m}, \boldsymbol{n}) \in \mathscr{P} \,|\, k_{g_n}(\boldsymbol{m})\, k_{h_n}(\boldsymbol{m}) > M\}. \end{aligned}$$

The latter set is, in the notation used in Theorem 7.1, P_n. So $R_n \times T_n \subset P_n$, so

$$\lambda(R_n \times T_n) \leqslant \lambda(P_n).$$

So $\mu(R_n)\,\mu(T_n) \leqslant \lambda(P_n) \leqslant 1$. But $\mu(R_n) \to 1$ and $\mu(T_n) \to 1$, so $\lambda(P_n) \to 1$ and $\lambda(P'_n) \to 0$.

Next, suppose that possibility (B) applies; $|g_n| < c$ for all n, and $|h_n| \to \infty$. Then $g_n = u_n a_n w_n$ gives $a_n = u_n^{-1} g_n w_n^{-1}$, so $|a_n| \leqslant 2|g_n| < 2c$, so $|a_n|^2 = 2\cosh t_n < 4c^2$. Hence for some $b > 0$, $0 \leqslant t_n < b$ for all n. Hence $e^{-b} \leqslant K_{g_n}(\boldsymbol{m}) < e^b$ for all $\boldsymbol{m} \in S^2$. Now let

$$V_n = \{\boldsymbol{n} \in S^2 \,|\, K_{h_n}(\boldsymbol{n}) > M^2 e^b\}.$$

Then, since $|h_n| \to \infty$, $\mu(V_n) \to 1$. Then, for each $(\boldsymbol{m}, \boldsymbol{n}) \in S^2 \times V_n$ we have

$$K_{g_n}(\boldsymbol{m}) > e^{-b}, \quad K_{h_n}(\boldsymbol{n}) > M^2 e^b$$

and hence $K_{g_n}(\boldsymbol{m})\,K_{h_n}(\boldsymbol{n}) > M^2$, giving

$$k_{g_n}(\boldsymbol{m})\,k_{h_n}(\boldsymbol{n}) > M.$$

Hence

$$S^2 \times V_n \subset P_n, \quad \lambda(S^2 \times V_n) \leqslant \lambda(P_n), \quad \mu(S^2)\,\mu(V_n) \leqslant \lambda(P_n)$$

so

$$\mu(V_n) \leqslant \lambda(P_n) \leqslant 1.$$

Since $\mu(V_n) \to 1$, $\lambda(P_n) \to 1$ and $\lambda(P'_n) \to 0$.

Finally, if possibility (C) applies, reversing the roles of g_n and h_n in the preceding paragraph again gives $\lambda(P_n) \to 1$ and $\lambda(P'_n) \to 0$. This establishes the required result in all possible cases.

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